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# POOLING, PRICING AND TRADING OF RISKS

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# Pooling, Pricing and Trading of Risks

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Abstract. Exchange of risks is considered here as a transferable-utility, cooperative game, featuring risk averse players. Like in competitive equilibrium, a core solution is determined by shadow prices on state-dependent claims. And like in finance, no risk can properly be priced only in terms of its marginal distribution. Pricing rather depends on the pooled risk and on the convolution of individual preferences. The paper elaborates on these features, placing emphasis on the role of prices and incompleteness. Some novelties come by bringing questions about existence, computation and uniqueness of solutions to revolve around standard Lagrangian duality. Especially outlined is how repeated bilateral trade may bring about a price-supported core allocation.

*Keywords*: cooperative game, transferable utility, core, risks, mutual insurance, contingent prices, bilateral exchange, supergradients, stochastic approximation.

## 1. Introduction

In a seminal paper Borch [4] considered risks as commodities and explored whether such items might be priced merely in terms of their marginal distributions or moments. Not surprisingly, his findings were negative. There can hardly exist a linear pricing regime of that sort. Further, even if existence were granted, price-taking exchange would not generally yield Pareto efficient allocations. And absent such efficiency, competitive equilibrium cannot obtain. In conclusion, Borch suggested that risk exchange had better be analyzed as a *cooperative game*.

This paper follows that suggestion. It extends work of Baton, Lemaire [2] and adds to Wilson's theory of syndicates [36]. Upon reconsidering Borch's approach, and a pioneering paper by Shapley and Shubik [33], a *transferable-utility,* cooperative game comes naturally on stage. It features agents who find it worthwhile to pool their risks [12], [13]. As customary, pooling smoothens nature's vagaries. Lucky agents can help unlucky ones; ups somewhere may mitigate downs elsewhere. In effect, when risk aversion is commonplace, and information is symmetric, the advantages of pooling suffice to render the *core* non-empty.

Thus, existence of a well defined solution is easily assured. Some queries remain though. *How is risk priced and shared*? Put differently*: how are premia and policies determined*? *Precisely where and how does risk aversion become crucial*? *What are*

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*the prospects for efficient computation? Can repeated trade eventually bring about stable solutions?*

The paper addresses all these issues. In doing so, much guidance comes from perfect equilibrium in exchange economies. Like there, pricing must be linear, behavior be price-taking, and any solution a member of the core. Unlike there, our solutions relate to saddle values instead of fixed points. Such values derive from standard Lagrangian duality. In short, rather than viewing risk exchange as a competitive market [6], [17], [19], [20], [26], [27], [31], it is seen here as a cooperative game with side payments [12], [13], [14], [30]. Provided payoffs be concave, such games are *balanced* - in fact, *totally balanced* - hence have nonempty cores [33]. Not surprisingly, risk-sharing then takes the form of mutual insurance.

Formally, the games at hand fit the frames of optimization and duality methods. Those frames entail considerable advantages. First, an optimum is usually easier to locate than a competitive equilibrium. Second, duality facilitates identification of precisely where and how risk aversion affects play. Third, existence and uniqueness hinge only upon absence of a duality gap and upon differentiability. Finally, risk trade can be, and often is, driven by bilateral exchanges.[1] Admittedly and regrettably, these many advantages are not for free: they obtain here because utility is transferable.

Section 2 sets the stage for cooperative risk sharing, addressed in Section 3. Appropriate sharing could come via a specific, price-generated core solution, as described in Section 4. Existence of such solutions - and possible uniqueness - is the concern of Section 5. Section 6 specializes to cases in which the parties agree on probabilities. Sections 7, 8 model repeated exchanges of risks, and Section 9 concludes.

**Notations** are as follows. Given a nonempty set $S$ and a vector space $V$, let $V^S$ denote the family of all functions from $S$ into $V$. For convenience, all vector spaces considered here below are finite-dimensional Euclidean. If $\mathbb{C} \subset V$ is nonempty closed convex, and $v \in V$, the *orthogonal projection* $P_{\mathbb{C}}v$ denotes the point in $\mathbb{C}$ closest to $v$. Let the player set $I$ have finite cardinality denoted $|I|$. When $(v^i) \in V^I$, it is often convenient to write $\vec{v}$ in lieu of $(v^i)$ and let $v^I := \sum_{i \in I} v^i$. The dual space $V^*$ comprises all continuous linear functions $v^* : V \to \mathbb{R}$. A function $f : V \to \mathbb{R} \cup \{-\infty\}$ is called *proper* if $dom f := \{v \in V : f(v) > -\infty\}$ is nonempty. Such a function admits a *conjugate* $f^* : V^* \to \mathbb{R} \cup \{+\infty\}$ defined by $f^*(v^*) := \sup \{f(v) - v^*(v) : v \in V\}$.[2]

## 2. THE SETTING

This section assembles pieces and parcels of the situation under scrutiny.

**A stochastic, two-period economy** is considered. Assets are traded right now, under uncertainty, and they yield returns next period. Ex ante, traders cannot precisely predict what state $s \in S$ will materialize next period. Ex post they all agree

[1] Bilateral transactions could proceed by means of predesigned contracts, such instruments then being insurance treaties.

[2] Replacing $f$ by $-f$ and $v^*$ by $-v^*$ gives the standard *Fenchel conjugate* [32]. Any proper, upper semicontinuous, concave $f$ is recovered via $f(v) = \inf \{f^*(v^*) + v^*(v) : v^* \in V^*\}$.

on which state has happened. The state space $S$ is an exhaustive, yet minimal list of mutually exclusive, economically relevant scenarios. For simplicity, take $S$ to be finite.[3]

**Players** constitute a fixed, finite set $I$. Agent $i \in I$ already owns a *risk* $\bar{y}^i = (\bar{y}^i_s)$ belonging to a linear subspace $Y$ of $\mathbf{Y} := \mathbb{E}^S$. The component $\bar{y}^i_s$ of his endowment is a commodity bundle in a finite-dimensional real Euclidean space $\mathbb{E}$. That component denotes his *claim*, indemnity or gross dividend in state $s$.[4] For reasons explained later, the players must contend with choices in $Y \subset \mathbf{Y}$. By way of example, while $\mathbf{Y}$ comprises *all* possible risks, $Y$ could be spanned by the prescribed risks $\bar{y}^i, i \in I$.[5]

**Preferences** are represented by *payoff functions* $\pi^i : \mathbf{Y} \to \mathbb{R} \cup \{-\infty\}$, satisfying $\pi^i(\bar{y}^i) > -\infty$. The objective $\pi^i(\cdot)$ of player $i$ might already be a reduced function. For example, if prior to asset trading, he must choose $x^i$ from some decision set $X^i$, with objective $\Pi^i : X^i \to \mathbb{R}$ and subject to $c^i(x^i) \geq \mathbf{y}^i \in \mathbf{Y}$, then posit

$$\pi^i(\mathbf{y}^i) := \sup \left\{ \Pi^i(x^i) : x^i \in X^i \text{ and } c^i(x^i) \geq \mathbf{y}^i \right\}. \quad (1)$$

Format (1) brings out *two* features. *First*, by tacit convention, $\pi^i(\mathbf{y}^i) = -\infty$ iff $\{x^i \in X^i : c^i(x^i) \geq \mathbf{y}^i\}$ is empty. More generally, the extreme value $-\infty$ serves to signal infeasibility. It accounts for restrictions and saves us explicit, repeated mention of these if any. *Second*, one cannot straightforwardly presume that payoff (1) be differentiable. Therefore, throughout the paper non-smooth payoffs are tolerated.

**Transferable utility** is presumed. In speaking rather of payoff, that entity is tacitly seen as cardinal, divisible, and transferable among agents. This assumption may be justified in two settings. For one, agent $i$ could be a *producer* who obtains monetary revenues $\pi^i(\mathbf{y}^i)$ from factor profile $\mathbf{y}^i \in \mathbf{Y}$. Alternatively, he might be a *consumer* who derives *quasi-linear utility* $\pi^i(\mathbf{y}^i) = y^i_0 + \pi^i_{-0}(\mathbf{y}^i_{-0})$ from a profile $\mathbf{y}^i = (y^i_0, \mathbf{y}^i_{-0})$ that has the amount $y^i_0 \in \mathbb{R}$ of "money" in some designated 0-th coordinate. The residual function $\pi^i_{-0}(\cdot)$ then reports the reservation price $\pi^i_{-0}(\mathbf{y}^i_{-0})$ that $i$ would assign to the accompanying commodity bundle $\mathbf{y}^i_{-0}$.

## 3. COOPERATIVE RISK SHARING

Any *coalition* or *consortium* $C \subseteq I$ of agents could aggregate their risks into $\bar{y}^C := \sum_{i \in C} \bar{y}^i$ and thereafter make transfers among themselves.[6] Motivation for such an

[3]The subsequent arguments can accommodate an infinite measure space $S$ together with the Hilbert space $\mathbf{Y} = \mathcal{L}^2(S, \mathbb{E})$ of square-integrable profiles $s \mapsto y_s \in \mathbb{E}$, mapping $S$ into a Euclidean space $\mathbb{E}$.

[4]That state-contingent claim could quite simply come as a financial credit or debit. Then $\mathbb{E} = \mathbb{R}$. Alternatively, if real assets generate various goods, mentioned on a finite list $G$, then $\mathbb{E} = \mathbb{R}^G$. More generally, any topological vector space $\mathbb{E}$ is applicable provided it be locally convex and Hausdorff. One can construe $\bar{y}^i$ as a consumption profile to which agent $i$ is entitled. This viewpoint fits to finance, and it opens up for inclusion of many time periods.

[5]In general, one would require that $Y$ be closed and complementable [25]. Given our finite-dimensional setting, $Y$ is automatically so. The particular instance $Y = \mathbf{Y}$ is referred to as *complete*.

[6]Nothing precludes that $i \in I$ already is a syndicate [36] or cartel, formed by agents of the same type. For such formation see [21] and [22].

enterprise might stem from $C$ contemplating potential payoff

$$\pi^C(\bar{y}^C) := \sup\left\{\sum_{i\in C}\pi^i(y^i) : \sum_{i\in C} y^i = \bar{y}^C \text{ with all } y^i \in Y\right\}. \tag{2}$$

Plainly, $\pi^C(\bar{y}^C) \geq \sum_{i\in C}\pi^i(\bar{y}^i) > -\infty$. A bliss value $\pi^C(\bar{y}^C) = +\infty$ makes no sense. So, *assume* $\pi^C(\bar{y}^C)$ *finite for all* $C \subseteq I$. In particular, agent $i$ obtains "autarky payoff" $\pi^{\{i\}}(\bar{y}^{\{i\}}) = \pi^i(\bar{y}^i)$ if he opts to avoid exchanges.

(2) is called a *sup-convolution*. It models pooling and friction-free redistribution of perfectly divisible risks.[7] Clearly, incentives for redistribution stem from complementarities or substitutions in the usage of technologies and endowments.

To incite everybody to join the *grand coalition* $C = I$, payoffs must be shared somehow. And sharing, for its viability, had better be efficient, incentive compatible, and "equitable". Any *core imputation* fills the bill. That solution concept, most popular in cooperative game theory, amounts here to specify a monetary compensation schedule which supports *Pareto efficiency* and *no blocking*:

**Definition 1.** (Core solutions) *A profile* $c = (c^i) \in \mathbb{R}^I$ *of side payments belongs to the* **core** *iff it entails*

$$\left.\begin{array}{lll} \textit{Pareto efficiency}: & \sum_{i\in I} c^i = \pi^I(\bar{y}^I) & \textit{and} \\ \textit{no blocking}: & \sum_{i\in C} c^i \geq \pi^C(\bar{y}^C) & \forall C \subset I. \end{array}\right\} \tag{3}$$

Pareto efficiency requires that total payoff be maximal and fully shared. No blocking means that each coalition receives, in sum, at least as much as when going alone. Is such a scheme of side payments available?[8] More precisely: can a core solution be exhibited, computed, interpreted and implemented? Yes, as seen next, if agents are risk averse, indeed it can!

## 4. Price-generated Core Solution

The subsequent arguments for viable collaboration proceed in terms of "price regimes" and standard Lagrangians. To introduce and conveniently handle these objects, equip the risk space $\mathbf{Y} = \mathbb{E}^S$ with a fixed inner product denoted by juxtaposition $\mathbf{y}^*\mathbf{y}$. Modulo that product, $\mathbf{Y}$ permits a decomposition $Y \oplus N$ into the direct sum of two orthogonal spaces, $N$ being the *normal complement* to the given closed subspace $Y$. This means that any $\mathbf{y} \in \mathbf{Y}$ comes as a unique sum $\mathbf{y} = y + n$ with $y \in Y, n \in N$, and $yn = 0$.

[7] For indivisible goods $g \in G$, see [34] and references therein. Then, if all risks come in integer amounts, one would use commodity space $\mathbb{E} = \mathbb{Z}^G$ where $\mathbb{Z} := \{0, \pm 1, \pm 2, ...\}$. Applying the *discrete convex analysis* - and notably the Fenchel-type duality theorem 5.2 in [28] - and presuming all payoff functions $\pi^i : \mathbf{Y} = \mathbb{Z}^{G\times S} \to \mathbb{R}\cup\{-\infty\}$ *M-concave*, it holds an analog of Theorem 1 below.

[8] It is known from [33] that concave payoffs $\pi^i$ and finite values $\pi^C(\bar{y}^C)$ suffice for the core to be nonempty. A fortiori, the game having *characteristic function* $S \supseteq C \mapsto \pi^C(\bar{y}^C) \in \mathbb{R}$ then becomes *totally balanced*. More is demanded here though: Some "specific" core element should "constructively" be brought to the fore; mere existence is not quite satisfactory.

Correspondingly, decompose the dual space $\mathbf{Y}^*$, which comprises all continuous linear functionals on $\mathbf{y}^* : \mathbf{Y} \to \mathbb{R}$, into the direct sum $Y^* \oplus N^*$ of $Y^* := N^\perp$ and $N^* := Y^\perp$; see [25]. Then, any $\mathbf{y}^* \in \mathbf{Y}^*$ equals a unique sum $\mathbf{y}^* = y^* + n^*$ with $y^* \in Y^*, n^* \in N^*$, and consequently, when $\mathbf{y} = y + n$, we get $\mathbf{y}^*\mathbf{y} = (y^* + n^*)(y + n) = y^*y + n^*n$.

Each payoff function $\pi^i : \mathbf{Y} \to \mathbb{R} \cup \{-\infty\}$ has a *conjugate* $\pi^{i*} : \mathbf{Y}^* \to \mathbb{R} \cup \{+\infty\}$. The latter, which records economic rent or consumer surplus, is defined by

$$\pi^{i*}(\mathbf{y}^*) := \sup \left\{ \pi^i(\mathbf{y}) - \mathbf{y}^*\mathbf{y} : \mathbf{y} \in \mathbf{Y} \right\}. \tag{4}$$

The function $\mathbf{y}^* \mapsto \pi^{i*}(\mathbf{y}^*)$ so constructed is lower semicontinuous and convex. For interpretation, regard $i$ as a producer who pays $\mathbf{y}^*\mathbf{y}$ for factor input $\mathbf{y}$, gets payoff $\pi^i(\mathbf{y})$, and collects profit $\pi^{i*}(\mathbf{y}^*)$. After these preparations associate the standard *Lagrangian*

$$L^C(\overrightarrow{\mathbf{y}}, \overrightarrow{\mathbf{y}}^*) := \sum_{i \in C} \left\{ \pi^i(y^i + n^i) + y^*(\bar{y}^i - y^i) - n^{i*}n^i \right\} \tag{5}$$

to problem (2). Here the grand vector $\overrightarrow{\mathbf{y}} := (\mathbf{y}^i)$ has components $\mathbf{y}^i = y^i + n^i \in \mathbf{Y}$, called *primal variables*, construed as inputs or consumption bundles. Similarly, $\overrightarrow{\mathbf{y}}^* := (\mathbf{y}^{i*})$ has components $\mathbf{y}^{i*} = y^* + n^{i*} \in \mathbf{Y}^*$, called *dual variables*, seen as prices. As expected from a Lagrangian, $\sup_{\overrightarrow{\mathbf{y}}} \inf_{\overrightarrow{\mathbf{y}}^*} L^C(\overrightarrow{\mathbf{y}}, \overrightarrow{\mathbf{y}}^*) = \pi^C(\bar{y}^C)$.

Problem (2) motivates formula (5) as follows: First, relax the balance requirement $\sum_{i \in C} y^i = \bar{y}^C$ of coalition $C$ by paying $\sum_{i \in C} y^*(y^i - \bar{y}^i)$ for a deviation $y^C - \bar{y}^C$. Second, relax the restriction $y^i \in Y$ by paying $n^{i*}n^i$ for a normal component $n^i \in N$. It follows from (4) and (5) that

$$\sup_{\overrightarrow{\mathbf{y}}} L^C(\overrightarrow{\mathbf{y}}, \overrightarrow{\mathbf{y}}^*) = \sum_{i \in C} \left\{ \pi^{i*}(\mathbf{y}^{i*}) + y^*\bar{y}^i \right\}.$$

**Definition 2.** (Shadow prices) $\overrightarrow{\mathbf{y}}^* := (\mathbf{y}^{i*})$, *with* $\mathbf{y}^{i*} = y^* + n^{i*} \in \mathbf{Y}^*$, *is declared a* ***shadow price regime*** *iff*

$$\pi^I(\bar{y}^I) \geq \sum_{i \in I} \left\{ \pi^{i*}(y^* + n^{i*}) + y^*\bar{y}^i \right\}. \tag{6}$$

Because the *gap*

$$\sum_{i \in I} \left\{ \pi^{i*}(y^* + n^{i*}) + y^*\bar{y}^i \right\} - \pi^I(\bar{y}^I)$$

always is nonnegative, a shadow price $\overrightarrow{\mathbf{y}}^* := (\mathbf{y}^{i*}) = (y^* + n^{i*})$ prevails iff the said gap is nil.

**Theorem 1.** (Shadow prices on risks generate core solutions) *For any shadow price the monetary payment profile* $c = (c^i)$, *with*

$$c^i := \pi^{i*}(y^* + n^{i*}) + y^*\bar{y}^i, \tag{7}$$

*belongs to the core. That is, it satisfies* (3).

**Proof.** For any coalition $C \subseteq I$ and price regime $\overrightarrow{\mathbf{y}}^*$ with $\mathbf{y}^{i*} = y^* + n^{i*}$, one has

$$\sum_{i \in C} \left\{ \pi^{i*}(y^* + n^{i*}) + y^* \bar{y}^i \right\} \geq \inf_{\overrightarrow{\mathbf{y}}^*} \sup_{\overrightarrow{\mathbf{y}}} L^C(\overrightarrow{\mathbf{y}}, \overrightarrow{\mathbf{y}}^*) \geq \sup_{\overrightarrow{\mathbf{y}}} \inf_{\overrightarrow{\mathbf{y}}^*} L^C(\overrightarrow{\mathbf{y}}, \overrightarrow{\mathbf{y}}^*) = \pi^C(\bar{y}^C). \tag{8}$$

Thus, invoking definition (7), the "no blocking constraints" in (3) are all easily satisfied. Clearly, this weak duality result tells that access to a competitive market can harm no coalition. But Pareto efficiency follows straightforwardly as well because, using shadow prices, the market clears. To wit,

$$\pi^I(\bar{y}^I) \geq \sum_{i \in I} \left\{ \pi^{i*}(y^* + n^{i*}) + y^* \bar{y}^i \right\} = \sum_{i \in I} c^i \geq \pi^I(\bar{y}^I).$$

The left-most inequality in the last string was assumed in (6), and the extreme right one derives for the instance $C = I$ from (8). $\square$

Theorem 1 inspires the hope that a core solution might be found - and implemented - in terms of a linear price regime. That regime depends on the entire preference profile and the aggregate risk.

Theorem 1 also brings out that existence of equilibrium *prices* can be discussed separately from that of *allocations*.[9] Put differently: existence of price-supported core imputations can be argued without reference to how risks are shared.

As in Negishi's approach to competitive equilibrium, individual preferences are aggregated into those of a single representative figure, here called the *convoluted* agent. As in finance, the premium alias the price of any insurance treaty is largely affected by how its indemnity co-varies with the aggregate risk.

The competitive, decentralized nature of shadow prices regime is speaking. If charged payment $y^*(y^i - \bar{y}^i) + n^{i*}n^i$ for replacing his endowment $\bar{y}^i + 0$ by $y^i + n^i$, agent $i$ would make a choice that perfectly fits problem (2) for the grand coalition $C = I$. Note that formula (7) pays him in two capacities: first, profit $\pi^{i*}(y^* + n^{i*})$ as a "producer" and second, reimbursement $y^* \bar{y}^i$ as a "claim-holder."

## 5. Existence and Uniqueness

Theorem 1 tells that a shadow price regime obtains iff it realizes the *saddle value* $\min \sup L^I = \sup \inf L^I$. Put differently: what comes to the fore is a lop-sided min-sup result. But, as is well known, existence of saddle values cannot generally be guaranteed unless some compactness, continuity and convexity conditions are in vigor. Ignoring compactness for a while, we are, as usual in microeconomics, left with concerns about *continuity* and *convexity of preferences.*

[9] Fenchel's duality theorem [5] facilitates that divorce. That is, dual problem attainment can discussed separately from primal attainment.

To appreciate these properties, and to understand the nature of shadow prices, consider a *convoluted agent,* who enjoys payoff

$$\pi^I(y^I, \vec{n}) := \sup \left\{ \sum_{i\in I} \pi^i(y^i + n^i) : \sum_{i\in I} y^i = y^I \text{ with all } y^i \in Y \right\}. \tag{9}$$

Two arguments affect this fictive fellow: first, an aggregate risk $y^I \in Y$ and second, a profile $\vec{n} = (n^i)$ of vectors $n^i \in N$, each normal to $Y$. Clearly, $\pi^I(y^I, \vec{n})$ equals the total payoff to grand coalition $I$ after replacing the prescribed $\bar{y}^I$ by $y^I \in Y$ and giving each agent $i$ a normal component $n^i \in N$. With minor abuse of notation, note that $\pi^I(\bar{y}^I, 0) = \pi^I(\bar{y}^I)$ where the right hand value was defined already in (2). More importantly, note that *if all $\pi^i$ are concave, then so is $\pi^I(\cdot,\cdot)$*. In other words: *if all agents are risk averse, so is the convoluted agent as well.*

Returning now to the issue of compactness, if

$$\pi^I(\cdot,\cdot) \text{ is finite-valued in a neighborhood of } (\bar{y}^I, 0) \in Y \times N^I, \tag{10}$$

and concave, then that concern is cared for. (10) says that if local perturbations of total endowments were possible around the reference point $(\bar{y}^I, 0)$, the aggregate payoff to the grand coalition would remain bounded.[10]

(10) yields existence and a "neoclassic", marginalistic interpretation of shadow prices. For the statement recall that $\mathbf{y}^* \in \mathbf{Y}^*$ is called a *supergradient* of a proper function $f : \mathbf{Y} \to \mathbb{R} \cup \{-\infty\}$ at the point $\mathbf{y}$, and we write $\mathbf{y}^* \in \partial f(\mathbf{y})$, iff $f(\mathbf{y}') \le f(\mathbf{y}) + \mathbf{y}^*(\mathbf{y}' - \mathbf{y})$ for all $\mathbf{y}' \in \mathbf{Y}$. Also recall that provided $f$ be concave and bounded below near $\mathbf{y}$, then its superdifferential $\partial f(\mathbf{y})$ is nonempty; see [24], [32]. In essence, that fact implies:

**Theorem 2.** (Existence and characterization of shadow price regimes)

• *Suppose the function $\pi^I(\cdot,\cdot)$ is concave. Then, under qualification* (10) *there exists a supergradient $(y^*, \vec{n}^*) \in \partial\pi^I(\bar{y}^I, 0)$, and it constitutes a shadow price regime $\overrightarrow{\mathbf{y}} = (\mathbf{y}^{i*})$ with $\mathbf{y}^{i*} = y^* + n^{i*}$.*

• *Conversely, any such shadow price regime generates a supergradient $(y^*, \vec{n}^*) \in \partial\pi^I(\bar{y}^I, 0)$.*

• *In sum, a shadow price regime, and a corresponding core solution* (7)*, can be defined iff $\pi^I(\cdot,\cdot)$ is superdifferentiable at $(\bar{y}^I, 0)$.*

• *If $\overrightarrow{\mathbf{y}}^* = (\mathbf{y}^{i*})$ with $\mathbf{y}^{i*} = y^* + n^{i*}$ is a shadow price regime, and $\overrightarrow{y} = (y^i)$ solves problem* (2) *for the grand coalition, then it holds for each $i$ that*

$$y^* + n^{i*} \in \partial\pi^i(y^i). \tag{11}$$

[10] (10) ensures that the pricing problem becomes inf-compact whence has a solution. Thus, while agents' choice sets may be unbounded - as in [19], [17], [27] and [31] - (10) eliminates problems caused by long or short positions.

**Proof.** As pointed out already, under qualification (10) the concave function $\pi^I(\cdot,\cdot)$, being finite-valued near $(\bar{y}^I, 0)$, has a supergradient $(y^*, \vec{n}^*)$ there. Further, $(y^*, \vec{n}^*) \in \partial\pi^I(\bar{y}^I, 0)$ iff

$$\pi^I(y^I, \vec{n}) + y^*(\bar{y}^I - y^I) - \vec{n}^*\vec{n} \le \pi^I(\bar{y}^I, 0) \text{ for all } (y^I, \vec{n}) \in Y \times N^I.$$

By (9) this holds iff

$$\sum_{i \in I} \left\{\pi^i(y^i + n^i) + y^*(\bar{y}^i - y^i) - n^{i*}n^i\right\} \le \pi^I(\bar{y}^I, 0) \text{ for all } (y^i, n^i) \in Y \times N. \quad (12)$$

This, in turn, amounts to have (6). Finally, for any dual optimal $(y^*, \vec{n}^*)$, take, on the left hand side of (12), the (partial) superdifferential with respect to $\mathbf{y}^i = y^i + n^i$ at any primal optimal $y^i$. This yields (11). $\square$

Concavity alias risk aversion played a crucial role in Theorem 2. To reduce that role to a minimum consider the smallest concave function $\hat{\pi}^I(\cdot,\cdot) \ge \pi^I(\cdot,\cdot)$. In tems of $\hat{\pi}^I$ two weaker hypotheses suffice for existence of a shadow price: first, $\hat{\pi}^I(\cdot,\cdot)$ should be superdifferentiable at $(\bar{y}^I, 0)$; second, one should have $\hat{\pi}^I(\bar{y}^I, 0) = \pi^I(\bar{y}^I, 0)$.

Thus risk aversion is really not needed, neither in small nor in large. Rather, what imports is concavity in the aggregate - and only with respect to the reference point $(\bar{y}^I, 0)$.

Inclusions (11) tell that all agents use the same $y^* \in Y^*$ to price choices within the "market space" $Y$.[11] That is, up to idiosyncratic normal components $n^{i*}, i \in I$, the players agree on *one* price in $Y$. In the *market game* [33] restricted to $Y$, any feasible exchange deemed desirable and affordable, will be made. The valuations of an *infeasible* $\mathbf{y} \in \mathbf{Y} \setminus Y$, one whose normal component $n$ does not vanish, will probably vary across agents.

If potential exchanges constitute a ***complete space***, that is, if $Y = \mathbf{Y}$, then clearly, all $n^{i*} = 0$, and things become simpler. In that instance $y^*$ is briefly named a *shadow price*, and it holds:

**Corollary.** (Shadow prices under completeness) *Suppose* $Y = \mathbf{Y}$.
• *If the function* $\pi^I(\cdot)$ *defined in* (2) *is finite-valued in a neighborhood of* $\bar{y}^I$ *and concave, then it is superdifferentiable at* $\bar{y}^I$*, and any supergradient* $y^* \in \partial\pi^I(\bar{y}^I)$ *constitutes a shadow price with all* $n^{i*} = 0$.
• *Conversely, any shadow price* $y^*$ *of that sort must satisfy* $y^* \in \partial\pi^I(\bar{y}^I)$.
• *In sum, a shadow price* $y^*$ *- and a corresponding core solution* (7) *- can be defined with all* $n^{i*} = 0$ *iff* $\pi^I(\cdot)$ *is superdifferentiable at* $\bar{y}^I$.
• *If* $y^*$ *is a shadow price, and* $(y^i)$ *solves problem* (2) *for the grand coalition, then for each* $i$

$$y^* \in \partial\pi^i(y^i). \quad \square$$

[11] Smooth versions of (11) are prominent in models of incomplete financial markets; see [26].

When merely one good comes into consideration, that is, when $\mathbb{E} = \mathbb{R}$, Wilson [36] lists several explicit solutions.

An extremal convolution like (2) has regularizing effects [7], [14]. In particular, this bears on possible uniqueness of shadow prices as stated next.

**Proposition 1.** (Unique shadow price) *Suppose the space is complete, payoffs $\pi^i(\cdot)$ are concave, and the optimal value $\pi^I(\bar{y}^I)$ is attained. Then, if at least one player $i$ has $\pi^i(\cdot)$ strictly concave and differentiable, the convoluted payoff function $\pi^I(\cdot)$ becomes differentiable at $\bar{y}^I$, and the shadow price is unique.*

**Proof.** The distinguished agent $i$ has a unique $y^i$ at which optimum is attained. By the last bullet here above $y^* = \nabla\pi^i(y^i)$. $\square$

It is often natural to assume $\pi^I$ monotone increasing in each component of $y^I$. Then $y^* \geq 0$. For illustration of Theorems 1 &2, and to expand instance (1), suppose

$$\pi^i(\mathbf{y}^i) := \sup\left\{\Pi^i(x^i, \mathbf{y}^i) : x^i \in X^i\right\}, \tag{13}$$

featuring a bivariate proper, concave function $\Pi^i(\cdot,\cdot)$ defined over a Euclidean vector space $X^i \times \mathbf{Y}$. Instance (1) obtains by setting $\Pi^i(x^i, \mathbf{y}^i) := \Pi^i(x^i)$ when $c^i(x^i) \geq \mathbf{y}^i$, and $-\infty$ otherwise. Coalition $C$ could then achieve

$$\pi^C(\bar{y}^C) := \sup\left\{\sum_{i\in C}\Pi^i(x^i, y^i) : \sum_{i\in C} y^i = \bar{y}^C,\ x^i \in X^i,\ y^i \in Y\right\}.$$

Let here $L^C := \sum_{i\in C}\left[\Pi^i(x^i, y^i + n^i) + y^*(\bar{y}^i - y^i) - n^{i*}n^i\right]$ and

$$\Pi^{i*}(x^*, \mathbf{y}^*) := \sup\left\{\Pi^i(x^i, \mathbf{y}^i) - x^*x^i - \mathbf{y}^*\mathbf{y}^i\right\}.$$

Note that

$$\sup_{\overrightarrow{(x^i, y^i, n^i)}} L^C = \sum_{i\in C}\left\{\Pi^{i*}(0, y^* + n^{i*}) + y^*\bar{y}^i\right\}.$$

Verbatim imitation of the proof of Theorem 1 yields:

**Proposition 2.** (Core solutions in terms of primitive payoff functions) *Given reduced payoff functions like* (13)*, suppose*

$$\pi^I(\bar{y}^I) \geq \sum_{i\in I}\left\{\Pi^{i*}(0, y^* + n^{i*}) + y^*\bar{y}^i\right\}$$

*for some price regime $\overrightarrow{\mathbf{y}}^* = (\mathbf{y}^{i*})$ with $\mathbf{y}^{i*} = y^* + n^{i*}$. Then, by offering agent $i$ compensation $c^i := \Pi^{i*}(0, y^* + n^{i*}) + y^*\bar{y}^i$, we get a core solution.* $\square$

While Theorem 2 addresses existence of shadow prices, separate arguments are required for the availability of equilibrium allocations. Proposition 4 in [12] yields:

**Proposition 3.** (Existence of equilibrium allocations) *Suppose the constrained, upper-level set*

$$\left\{\vec{y} = (y^i) \in Y^I : \sum_{i\in I} y^i \in \mathcal{K},\ \sum_{i\in I} \pi^i(y^i) \geq r\right\} \text{ is compact} \tag{14}$$

*for every compact $\mathcal{K} \subset Y$ and every real $r$. Then $\pi^I$ becomes upper semicontinuous proper, and the value $\pi^I(\bar{y}^I)$ will be attained by some feasible allocation $(y^i)$. If moreover, each $\pi^i$ is lower semicontinuous at the corresponding $y^i$, then $\pi^I$ becomes continuous at $\bar{y}^I$.* $\square$

The hypotheses in Proposition 3 serve to compactify the attractive part of the aggregate decision set. A similar proposition allows relaxed profiles $\overrightarrow{\mathbf{y}} = (\mathbf{y}^i) \in \mathbf{Y}^I$. The continuity of $\pi^I$ at $\bar{y}^I$ ensures its superdifferentiability at that point. Thus (14) relates to (10). Broadly, the feasible allocations that provide sufficient aggregate payoff, must be bounded.

## 6. Common Predictions and Separable Preferences

Assume henceforth that everybody holds the same opinion about the likelihood of various outcomes.[12] Formally, there is a common strictly positive probability distribution $p = (p_s)$ over $S$. Each linear functional $\mathbf{y}^*$ on $\mathbf{Y}$ can now be represented in terms of the statistically motivated, probabilistic inner product $\mathbf{y}^*\mathbf{y} := \sum_{s\in S} p_s y_s^* y_s$ with $y_s^*, y_s \in \mathbb{E}$. Such representation is particularly useful for the important instance where preferences are of von Neumann-Morgenstern separable type. Ex ante payoff

$$\pi^i(\mathbf{y}^i) := E\pi^i_\bullet(\mathbf{y}^i_\bullet) = \sum_{s\in S} p_s \pi^i_s(y^i_s) \tag{15}$$

then equals the expected value of its ex post state-dependent counterpart, and (11) amounts to have $y_s^* + n_s^{i*} \in \partial\pi^i_s(y^i_s)$ for each $s$. Given separable format (15), if the members of coalition $C$ pool their claims ex post, *after* $s$ has been unveiled, having then available the aggregate $\bar{y}^C_s := \sum_{i\in C} \bar{y}^i_s$, it might, in that circumstance, "shoot for" over-all contingent payoff

$$\pi^C_s(\bar{y}^C_s) := \sup\left\{\sum_{i\in C} \pi^i_s(y^i_s) : \sum_{i\in C} y^i_s = \bar{y}^C_s,\ y^i_s \in \mathbb{E}\right\}. \tag{16}$$

Thus, one may speak about contingent, *state-dependent cooperation*, implemented after $s$ has become manifest. Like before, a compensation scheme $c_s = (c^i_s) \in \mathbb{R}^I$

[12] Admittedly, it is somewhat unsatisfactory that information is presumed symmetric. As known from [37], [23] and other studies, asymmetries can eliminate good opportunities for mutual insurance. Also, communication raises the prospect of additional elimination.

belongs to the core of the cooperative game that prevails in state $s$ iff

$$\begin{array}{ll} \text{Pareto efficiency obtains:} & \sum_{i\in I} c_s^i = \pi_s^I(\bar{y}_s^I), \quad \text{and} \\ \text{there is no blocking:} & \sum_{i\in C} c_s^i \geq \pi_s^C(\bar{y}_s^C) \quad \forall C \subset I. \end{array}$$

Also, like before, if $y_s^* \in \mathbb{E}^*$ is a Lagrange multiplier of problem (16) for $C = I$, then by providing compensation

$$c_s^i := \pi_s^{i*}(y_s^*) + y_s^* \bar{y}_s^i$$

to agent $i$, one obtains a core solution ex post in state $s$.

Opportunistic behavior of this sort - where agents prefer to *wait and see,* where different realizations are treated apart from each other - will not generally fit with (3). The simple reason is, of course, that in passing from (3) to (16) all constraints $y^i \in Y$ were dropped or ignored. When relieved of his constraint, agent $i$ receives expected compensation $\bar{c}^i := \sum_{s\in S} p_s c_s^i$. To emphasize the role of $Y$ in (2) let $\pi^I(\bar{y}^I, Y)$ denote the optimal value for the grand coalition $C = I$ there. Clearly, when $Y \subset \mathbf{Y}$, it holds that

$$\sum_{i\in I} \bar{c}^i = \pi^I(\bar{y}^I, \mathbf{Y}) \geq \pi^I(\bar{y}^I, Y) = \sum_{i\in I} c^i.$$

If $Y \subsetneq \mathbf{Y}$, the last inequality tends to be strict. Equality holds however, under completeness:

**Theorem 3.** (Completeness of the market and time consistency of cooperation) *Suppose claims can be traded in a complete space; that is, suppose $Y = \mathbf{Y}$. Then any shadow price $y^* = (y_s^*)$ supports an over-all ex ante core solution $c^i := \pi^{i*}(y^*) + y^*\bar{y}^i$ as well as an ex post, contingent core solution $c_s^i := \pi_s^{i*}(y_s^*) + y_s^*\bar{y}_s^i$ in each state $s$. It holds that $c^i = \sum_{s\in S} p_s c_s^i$. And it does not matter whether these cooperative treaties were written before or after the state has been unveiled.* $\square$

Under completeness and separable preferences (15), if for each state $s$ the convoluted payoff $\pi_s^I(\cdot)$, as defined in (16), is differentiable at $\bar{y}_s^I$, then the price (insurance premium) of $y$ equals $\nabla\pi_s^I(\bar{y}_s^I)y_s$ ex post and $\sum_{s\in S} p_s \nabla\pi_s^I(\bar{y}_s^I)y_s$ ex ante.

It appears that some diversity in probability assessments may be accommodated. Specifically, if agent $i$ believes state $s$ will materialize with probability $f_s^i p_s$, simply replace $\pi_s^i$ by $f_s^i \pi_s^i$ and proceed with the same analysis as here above. In other words, the assessments could all be absolutely continuous with respect to a common probability measure $(p_s)$.

## 7. Bilateral Exchange of Risks

Construction (2) invites some pressing questions. Namely, when $C = I$, who undertakes the optimization and how? Further, since an efficient solution seemingly requires revelation of true preferences, can it be implemented? May parties fall victim to strategic communication?

To separate these issues consider first how a center or consultant, who holds all necessary information, might take up the computational task. For that purpose *suppose problem* (2) *admits an optimal solution when* $C = I$. Also, for simplicity, *suppose hereafter that each payoff function* $\pi^i$ *be finite-valued and concave across the entire space* $\mathbf{Y}$.[13] Thus, there are no implicit restrictions besides the explicit constraint $y^i \in Y$. Moreover, again for simplicity, *suppose all superdifferentials* $\partial\pi^i$ *are uniformly bounded.*[14] Let $\{\gamma_k\}$ be a numerical sequence of so-called *step sizes*, selected a priori subject to

$$\gamma_k > 0, \quad \sum_{k=0}^{\infty} \gamma_k = +\infty, \quad \text{and} \quad \gamma_k \to 0. \tag{17}$$

The computing center, or the said consultant, could proceed by iterated **gradient projection,** described as follows:

• *Start* at stage $k := 0$ with step size $\gamma := \gamma_0$ and choices $y^i \in Y, i \in I$, determined by history, guesswork or accident. It should hold though, that $\sum_{i \in I} y^i = \bar{y}^I$.

• *Select* for each agent $i$ a marginal payoff vector $M^i \in \partial\pi^i(y^i) \subset \mathbf{Y}$. Project these onto the subspace $Y$ to have $m^i := P_Y M^i$ with mean $\bar{m} := \sum_{i \in I} m^i / |I|$.

• *Update* choices by the rule

$$y^i \leftarrow y^i + \gamma(m^i - \bar{m}) \text{ for all } i. \tag{18}$$

• *Move* to the next stage $k \leftarrow k + 1$ with new step size $\gamma \leftarrow \gamma_k$.

• *Continue* to *Select* until convergence. □

**Proposition 4.** (Coordinated convergence to the core) *The described procedure of iterated gradient projection converges to an optimal solution of problem* (2) *for the grand coalition. Moreover, such a solution generates a shadow price regime.*

**Proof.** First, consider the problem to find the best approximate of any given vector $(\hat{\mathbf{y}}^i) = (\hat{y}^i + \hat{n}^i) \in \mathbf{Y}^I$ in the affine subspace $\mathcal{Y} := \left\{(y^i) \in Y^I : \sum_{i \in I} y^i = \bar{y}^I\right\}$. Formally, this amounts to minimize $\sum_{i \in I} \|y^i - \hat{\mathbf{y}}^i\|^2$ s.t. $\sum_{i \in I} y^i = \bar{y}^I$ and all $y^i \in Y$. Since $\|y^i - \hat{\mathbf{y}}^i\|^2 = \|y^i - \hat{y}^i\|^2 + \|\hat{n}^i\|^2$, one gets $y^i = \hat{y}^i + (\bar{y}^I - \hat{y}^I)/|I|$.

Given now $y^i \in Y$ for all $i \in I$, and also $\sum_{i \in I} y^i = \bar{y}^I$, when taking the projection $P_{\mathcal{Y}}$ of the grand vector $\overrightarrow{\mathbf{y}} + \gamma\overrightarrow{\mathbf{M}}$: $= [y^i + \gamma M^i]_{i \in I} \in \mathbf{Y}^I$ onto $\mathcal{Y}$, we get, by the above observation, as closest approximation the point $[y^i + \gamma(m^i - \bar{m})]_{i \in I}$. Thus, iteration

[13] When $\pi^i$ is bounded above, this holds if $\pi^i$ is replaced by

$$\hat{\pi}^i(\mathbf{y}) := \sup\left\{\pi^i(\mathbf{y}') - C^i(\mathbf{y} - \mathbf{y}') : \mathbf{y}' \in \mathbf{Y}\right\},$$

using a function $C^i : \mathbf{Y} \mapsto \mathbb{R}_+$ which is convex and vanishes only at the origin. In particular, when $C^i = \|\cdot\|^2$, the so regularized function $\hat{\pi}^i$ becomes smooth as well; see [7] Thm. 5.1.

[14] This holds if there exist $r, \rho > 0$ and an optimal solution $(\hat{y}^i)$ to (2) with $C = I$, such that $\sum_{i \in I} \|\hat{y}^i - y^i\| \geq r$ implies $\sum_{i \in I} \left\{\pi^i(\hat{y}^i) - \pi^i(y^i)\right\} \geq \rho$. In particular, some degree of strong concavity in all objectives would suffice.

(18) is nothing else than the method of (super-)gradient projection

$$\overrightarrow{\mathbf{y}} \leftarrow P_{\mathcal{Y}}\left[\overrightarrow{\mathbf{y}} + \gamma\overrightarrow{\mathbf{M}}\right]$$

applied to problem (2) with $C = I$. Because that problem is concave, and because by assumption it admits an optimal solution, convergence follows from received theory; see [10]. After convergence to an optimal profile $\overrightarrow{\mathbf{y}} = (y^i)$, pick a common $y^* \in \cap_{i \in I} P_Y\left[\partial \pi^i(y^i)\right]$, and for each agent $i$, a normal $n^{i*} \in \partial \pi^i(y^i) - y^*$. Then (11) holds. In particular, if each payoff $\pi^i(\cdot)$ is differentiable in classical Gâteaux sense at $y^i$, it holds that $y^* = P_Y\left[\partial \pi^i(y^i)\right]$ and $n^{i*} = \partial \pi^i(y^i) - y^*$. $\square$

V. Pareto regarded economic markets as *decentralized computing machines.* Subscribing here to his view, the centralized algorithm, just described, had better be replaced by a non-coordinated process driven by the agents themselves. The one proposed next is of that preferred sort. It purports to reflect **repeated bilateral exchanges of risk**, proceeding as follows:

• *Start* at stage $k := 0$ with step size $\gamma := \gamma_0$ and choices $y^i \in Y, i \in I$, determined by history or accident such that $\sum_{i \in I} y^i = \bar{y}^I$.
• *Choose* two agents $i, i' \in I$ independently of past choices and according to a uniform probability. Pick $m^i \in P_Y \partial \pi^i(y^i)$, $m^{i'} \in P_Y \partial \pi^{i'}(y^{i'})$.
• *Update* the choices of only these two agents by

$$y^i \leftarrow y^i + \gamma(m^i - m^{i'}) \ \ \textit{and} \ \ y^{i'} \leftarrow y^{i'} + \gamma(m^{i'} - m^i). \tag{19}$$

• *Move* to the next stage $k \leftarrow k+1$ with new step size $\gamma \leftarrow \gamma_k$.
• *Continue* to *Choose* two agents until convergence. $\square$

**Theorem 4.** (Convergence of decentralized, bilateral exchanges) *Suppose repeated bilateral exchanges of risks produces uniformly bounded sequences of variates $y^i$, $m^i$. Besides* (17) *suppose $\sum \gamma_k^2 < +\infty$. Then the sequence of $y^i$ converges to an optimal solution of problem* (2) *for the grand coalition. Moreover, such a solution generates a shadow price regime.*

**Proof.** If merely two agents are around, the setting is quite as in Proposition 4. So, with no loss of generality, posit $I = \{1, ..., |I|\}$ with $|I| > 2$. Let the random event space $\Omega$ consist of all ordered pairs $(i, i') \in I \times I$. Outcome $\omega = (i, i')$ means that players $i, i' \in I$ are sampled (with replacement) and offered the opportunity to trade risks between themselves. Quite naturally, such opportunities should emerge in egalitarian manner. Accordingly, endow $\Omega$ with the uniform probability measure; that is, each $\omega$ is selected with probability $1/|I|^2$.

Take expectation with respect to $\omega$ in $y^i + \gamma(m^i - m^{i'})$ to get $y^i + \gamma(m^i - \bar{m})/|I|$. Thus, in expectation (19) amounts to a scaled down version of (18). Since the latter converges, so does the bilateral process by Prop. 2.1 in [3]. □

It deserves emphasis that players may very well be scantly informed. Trade can, in principle, be postponed until shadow prices prevail. If so, processes (18) and (19) both follow the long tradition of Walrasian tâtonnement [11], [29]. They differ however, from most received versions - and from Wilson's budgetary adjustment [35] - in dispensing with explicit listing of prices. For greater realism transactions may happen in the interim. Then, some trades, made before prices settle at equilibrium levels, may be regretted by some party later on. Other trades may have turned out favorable when viewed in hindsight.

## 8. TRADE OF INSURANCE TREATIES

It is time at last, to justify why only risks residing in a subspace $Y \subset \mathbf{Y}$ are traded. Clearly, if any exchange in $\mathbf{Y}$ were possible, the setting would be that of a barter economy.

For a realistic optic, one that fits insurance and finance, suppose exchange is mediated only via a finite set $J$ of so-called instruments, briefly referred to as *insurance treaties.* Each $j \in J$ is a contract that promises to pay its holder a specified indemnity, coverage or gross dividend $d_{sj} \in \mathbb{E}$ if state $s \in S$ comes about. Suppose treaties are *perfectly divisible* and traded without quantity restrictions and transaction costs.

As before, there are only two time periods: *now* and *later.* This means that all treaties expire after one appropriately defined time-step. By a *portfolio* is understood a vector $x = (x_j) \in X := \mathbb{R}^J$, saying precisely how much is held of various policies/contracts. Note that portfolio $x$ yields indemnity $y_s = \sum_{j\in J} d_{sj}x_j$ in state $s$. So, letting $D = [d_{sj}]$ denote the $S \times J$ indemnity (dividend) matrix, portfolio $x$ entitles its holder to the profile $y = Dx$.

Correspondingly, let $Y := \operatorname{Im} D := DX := \{Dx : x \in X\}$ be the image space of $D$, spanned by its columns $d_{\cdot j} \in \mathbf{Y}, j \in J$. The possibly strict subspace $Y \subset \mathbf{Y} = \mathbb{E}^S$ consists of marketable indemnity profiles. Vectors in $\mathbf{Y} \diagdown Y$ cannot be synthesized via the given instruments; they are not "in the market." A profile $y \in Y$ will be realized by any portfolio $x \in X$ which solves $Dx = y$. At least one such $x$ exists by the definition of $Y$. Uniqueness of $x$ follows iff $D : X \to Y$ is one-to-one. In that case $|J| = rank(D) = \dim Y$. In particular, when $Y = \mathbb{E}^S$, there must be as many treaties as there are states.

At the outset agent $i$ holds portfolio $\bar{x}^i$, generating risk $\bar{y}^i := D\bar{x}^i$. Coalition $C$, holding the aggregate $\bar{x}^C := \sum_{i\in C} \bar{x}^i$, can achieve

$$\begin{aligned}
\pi^C(\bar{y}^C) &= \sup\left\{\sum_{i\in C} \pi^i(y^i) : \sum_{i\in C} y^i = \bar{y}^C := D\bar{x}^C,\ y^i = Dx^i, x^i \in X\right\} \\
&= \sup\left\{\sum_{i\in C} \pi^i(Dx^i) : \sum_{i\in C} Dx^i = D\bar{x}^C, x^i \in X\right\}.
\end{aligned}$$

This instance fits Proposition 3. Note that unlimited shortselling is allowed. Otherwise appropriate constraints $x^i \in X^i \subset X$ would be imposed. The transpose matrix $D^*$ transports risk prices $\mathbf{y}^* \in \mathbf{Y}^*$ back to prices $x^* = D^*\mathbf{y}^*$ on underlying portfolios by the rule $x_j^* = \sum_{s\in S} y_s^* d_{sj}$. Accordingly, posit $\hat{\pi}^i(x) := \pi^i(Dx)$ to have $\hat{\pi}^{i*}(D^*\mathbf{y}^*) = \pi^{i*}(\mathbf{y}^*)$. Because $D^*\mathbf{y}^* = 0$ if $\mathbf{y}^* \in (\operatorname{Im} D)^\perp = Y^\perp = N^*$, we get

**Proposition 5.** (Shadow prices on treaties generate core solutions) *For any inital holding profile* $(\bar{x}^i)$ *and associated shadow price regime* $\overrightarrow{\mathbf{y}}^* = (\mathbf{y}^{i*}) = (y^* + n^{i*})$ *on risks, there is a corresponding price regime* $x^* := D^*\mathbf{y}^{i*} = Dy^*$ *on portfolios such that the payment scheme*

$$c^i := \pi^{i*}(y^* + n^{i*}) + y^*(D\bar{x}^i) = \pi^{i*}(y^* + n^{i*}) + (D^*y^*)\bar{x}^i = \hat{\pi}^{i*}(x^*) + x^*\bar{x}^i \qquad (20)$$

*belongs to the core. That is, it satisfies* (3). □

Clearly, $i$ could have access to a particular set $J^i$ of treaties, defined by an $S \times J^i$ matrix $D^i$. If so, (20) would remain a core solution with $D^i$ instead of $D$. Agent $i$ might also have handy a technology by which his *effort* $e^i$ produces a profile $\mathcal{E}^i(e^i) \in Y$. Then, if $i$ enjoys concave payoff $\Pi^i(e^i, y^i)$, coalition $C$ gets payoff

$$\pi^C(\bar{y}^C) = \sup\left\{\sum_{i\in C} \Pi^i(e^i, D^i x^i + \mathcal{E}^i(e^i)) : \sum_{i\in C}\left[D^i x^i + \mathcal{E}^i(e^i)\right] = \bar{y}^C\right\}.$$

When however, only agent $i$ knows $e^i$ or $\mathcal{E}^i(\cdot)$, there may be problems with hidden actions or types, these making the prospects for efficient cooperation appear less good.[15]

This section ends by considering **repeated bilateral exchanges of portfolios**. It could go as follows:

- *Start* at stage $k := 0$ with step size $\gamma := \gamma_0$ and choices $x^i \in X, i \in I$, determined by history or accident.
- *Choose* two agents $i, i'$ according to the uniform distribution (i.e. in equiprobable manner).
- *Select* marginal payoffs $m^i \in \partial\pi^i(Dx^i)$, $m^{i'} \in \partial\pi^{i'}(Dx^{i'})$ and let $x^{i*} = D^*m^i$, $x^{i'*} = D^*m^{i'}$.
- *Update* the choices by bilateral exchange of portfolios

$$x^i \leftarrow x^i + \gamma(x^{i*} - x^{i'*}) \quad and \quad x^{i'} \leftarrow x^{i'} - \gamma(x^{i*} - x^{i'*})$$

- *Move* to next stage $k \leftarrow k+1$ with new step size $\gamma \leftarrow \gamma_k$.
- *Continue* to *Choose* two agents until convergence. □

[15] Studies dealing with core solutions under asymmetric information include [8], [9], [23] and [37].

Like Theorem 3 one proves:

**Theorem 5.** *Under the assumption of Theorem 4 repeated bilateral exchanges of portfolios lead to an optimal solution of problem* (2) *for the grand coalition.* □

## 9. Concluding Remarks

Cooperation, exchange and trade often appear as cousins in economics. Here, given symmetric information and wide-spread risk aversion, or at least risk neutrality, the cooperative incentives become so strong and well distributed that the grand coalition can safely form. Its formation means that all risks are pooled and that benefits be shared in ways not blocked by any subgroup. However, when preferences are not convex, the price-based compensation scheme (7) is likely to reside out-of-core; see [12], [14]. It appears though, that if all risk holders were negligible, then convexity could be dispensed with; see [1], [38].

It seems worthwhile to allow more time periods and explore problems related to time-consistency. Then, for properties of the characteristic function and the core, see [18].

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