

Submodular functions, generalized polyhedra, conforming preorders, and cointeracting bialgebras

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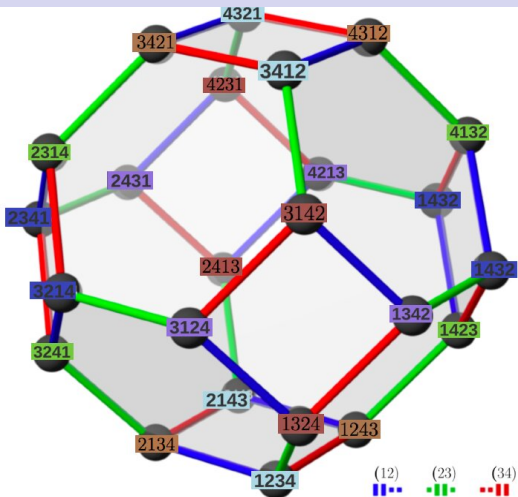
November 29, 2024

Highlights

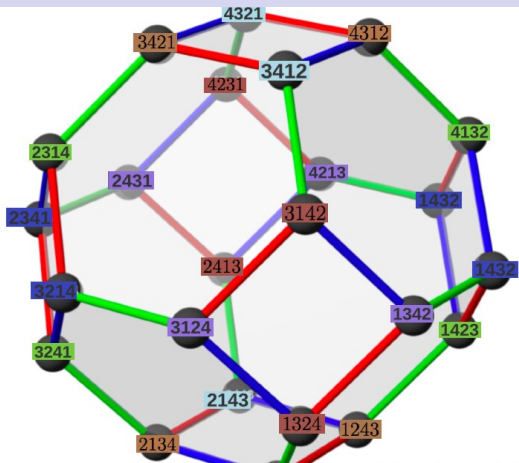
Polarizations

- 1 Generalized permutahedra
- 2 Normal fans
- 3 Submodular functions
- 4 Conforming preorders
- 5 Cointeracting bialgebras

Three-dimensional permutahedron



Three-dimensional permutahedron



Maximal faces

Ordered partitions (S_1, S_2) of $\{1, 2, 3, 4\}$.

Two-dimensional permutahedron

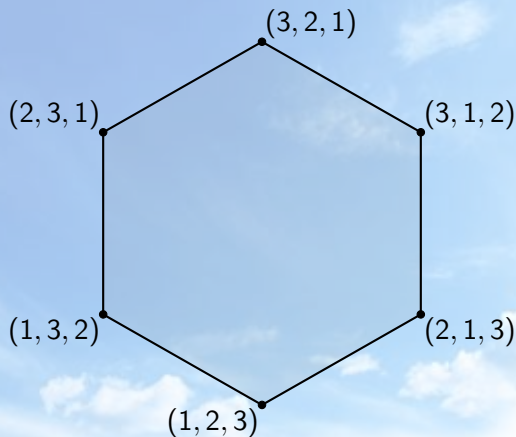
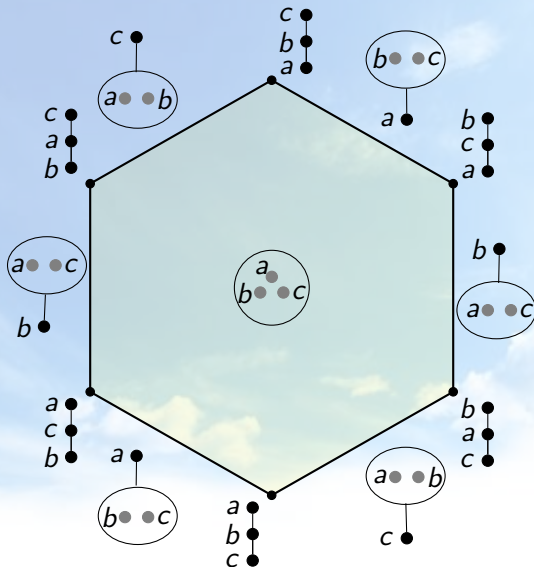


Figure: Two-dimensional permutahedron

Associated preorders



Preorders

Definition

A relation \leq on a set X is a **preorder** on X if it is:

1. **Reflexive:** $x \leq x$ for $x \in X$.
2. **Transitive:** For $x, y, z \in X$ then:

$$x \leq y \text{ and } y \leq z \text{ implies } x \leq z.$$

Preorders with two elements

$V = \{a, b\}$, two distinct elements

	$a \leq b$	$b \leq a$
1.	NO	NO
2.	YES	NO
3.	NO	YES
4.	YES	YES

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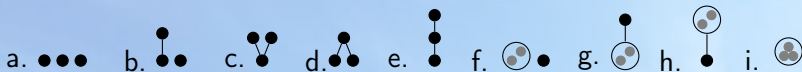
	$a \leq b$	$b \leq a$
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Pre(V):

Isomorphism classes:

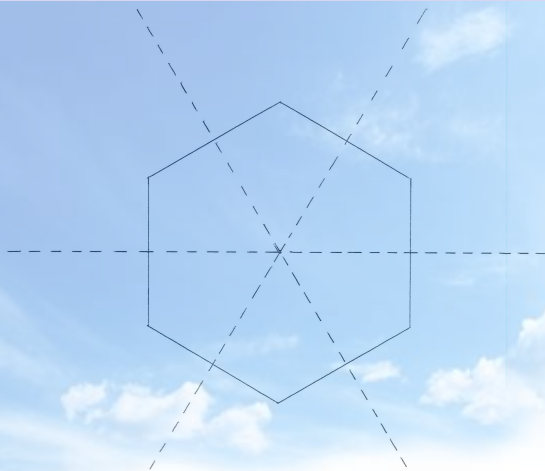


Preorders with three elements



Normal fan

Hexagon



Braid fan

- The hyperplanes $x_i = x_j$ divided \mathbb{R}^n into chambers.
- It divides $\mathbb{R}^n / \mathbb{R}(1, 1, \dots, 1)$ into pointed chambers.
- The faces of all these chambers gives a fan: [the braid fan](#)

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Definition: **Generalized permutahedron**

A convex polytope whose normal fan is a **coarsening** of the braid fan

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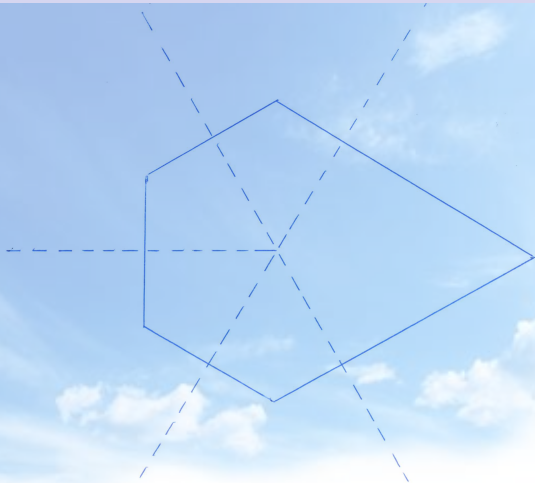
A convex polytope whose normal fan is a [coarsening](#) of the braid fan

Examples

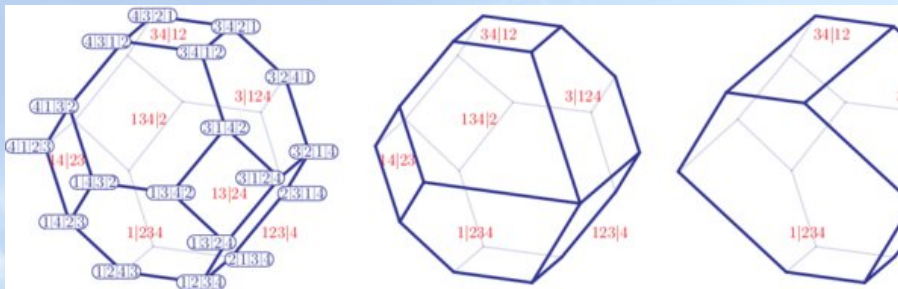
Associahedra, cyclohedra, graphohedra, nestohedra, zonotopes, ...

Pentagon

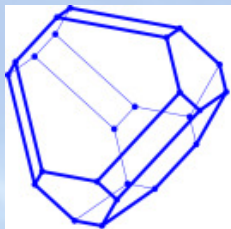
As generalized permutahedron



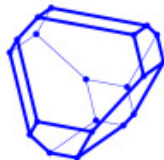
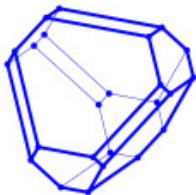
Three-dimensional generalized permutahedra



Three-dimensional generalized permutahedra



permutahedron



cyclohedron



associ

Submodular function

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For all subsets A, B of I :

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For any subsets $A \subseteq B \subseteq I$ and $S \subseteq I \setminus B$:

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- Much used in economics, game theory, and optimization
- In minimization problems: Model notions of **complexity, similarity and cooperation**.
- In maximization problems: Model notions of **diversity, information and coverage**.

Matroids and polymatroids

- Submodular function $z : P(I) \rightarrow \mathbb{N}$.

Polymatroidal:

- Grounded: $z(\emptyset) = 0$,
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Correspond precisely to the rank functions of matroids.

Generalized permutahedra and submodular functions

Submodular function $z : P(I) \rightarrow \mathbb{R}$ with $z(\emptyset) = 0$.

\rightsquigarrow convex polytope in \mathbb{R}^I defined by:

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Bijection

Generalized permutahedra in \mathbb{R}^I

\updownarrow 1-1

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Generalized permutahedra and submodular functions

Three-dimensional permutahedron

$$z : P(\{a, b, c, d\}) \rightarrow \mathbb{R}$$

given by

$$z(I) = \begin{cases} 4 & |I| = 1, \\ 4 + 3 & |I| = 2, \\ 4 + 3 + 2 & |I| = 3, \\ 4 + 3 + 2 + 1 & |I| = 4. \end{cases}$$

Three-dimensional associahedron (pentagon)

$$z : P(\{a, b, c\}) \rightarrow \mathbb{R}$$

given by

(-) (I) (-) 3 (-I) (Ia) 5 (-a) (-ba)

Restrictions and contractions

Let $S \subseteq I$

Restriction:

$$z|_S : P(S) \rightarrow \mathbb{R}, \quad z|_S(U) = z(U).$$

Contraction:

$$z/_S : P(I \setminus S) \rightarrow \mathbb{R}, \quad z/_S(U) = z(S \cup U) - z(S).$$

For $A \subseteq B \subseteq I$, write $C = B \setminus A$

$$z_{B \setminus A} := (z|_B)/_A = (z/_A)|_{(B/A)}.$$

Products

$$z : P(S) \rightarrow \mathbb{R}, \quad w : P(T) \rightarrow \mathbb{R}.$$

Their **product** is the submodular function:

$$\begin{aligned} z \cdot w &: P(S \cup T) \rightarrow \mathbb{R} \\ z \cdot w(E) &= z(E \cap S) + z(E \cap T) \end{aligned}$$

Submodular functions and preorders

Submodular functions $z : P(I) \rightarrow \mathbb{R} \cup \{\infty\}$.

The set of all $S \subseteq I$ with $z(S)$ finite, is a finite topology.

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Finite topologies $\xleftrightarrow{1-1}$ finite preorders

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Submodular functions $z : P(I) \rightarrow \{0, \infty\}$

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Preorders on I

Conforming preorders

Submodular functions give an **overarching common** framework for:

- Matroids
- Preorders

Conforming preorders

Submodular functions $z : P(I) \rightarrow \mathbb{R} \cup \{\infty\}$

$\updownarrow 1 - 1$

Extended generalized permutahedra (EGP) $\Pi(z)$

These are polyhedrons (unbounded if z takes values ∞) whose normal fan is a coarsening of a subfan of the braid arrangement.

Conforming preorders

P preorder, $A \subseteq B$ down-sets of P . Difference $C := B \setminus A$ is **convex subset** of P .

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if and only if
 $z' := z_{B \setminus A}$ decomposes as a product

$$z' = z'_{|C_1} \cdot z'_{|C_2}.$$

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Then: $z_{B/A}$ only depends on C . We write $z_C := z_{B \setminus A}$.

Conforming preorders

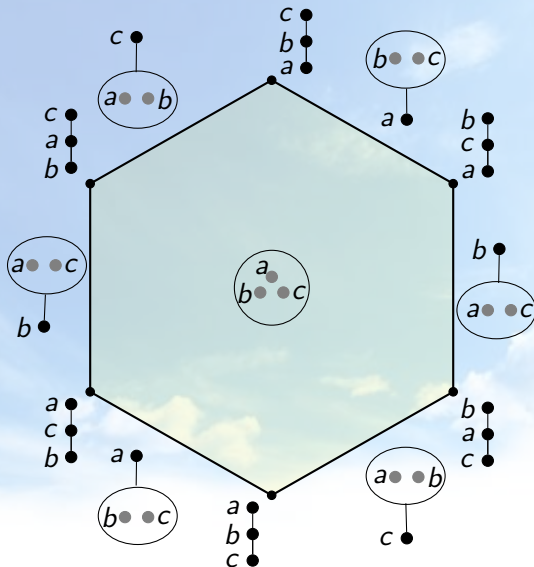
Bijection theorem

Faces of EGP $\Pi(z)$

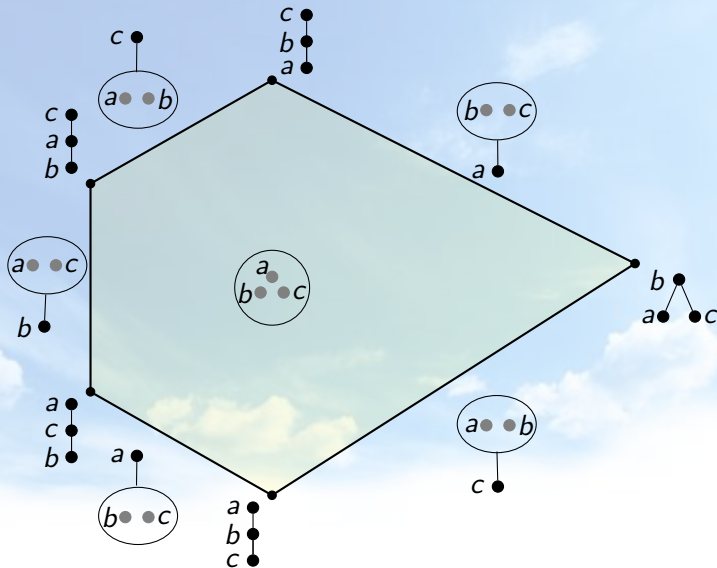
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Preorders P conforming to z

Associated preorders



Associated preorders



Preorders and faces

History

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- Morton, Pachter, Shiu, Sturmfels, Wienand, *Convex rank tests and semi-graphoids*, SIAM Journal of Discrete Mathematics, 2009.

Faces and conforming preorders

Let the preorder P conform to z . It corresponds to a face F of $\Pi(z)$.

Submodular function

$$z_P := \prod_{C \text{ bubble in } P} z_C.$$

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Fact: The extended generalized permutahedron $\Pi(z_P) = F$.

Faces and conforming preorders

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Fact: z^P is the submodular function corresponding to the cone of $\Pi(z)$ at F .

Cone modular function

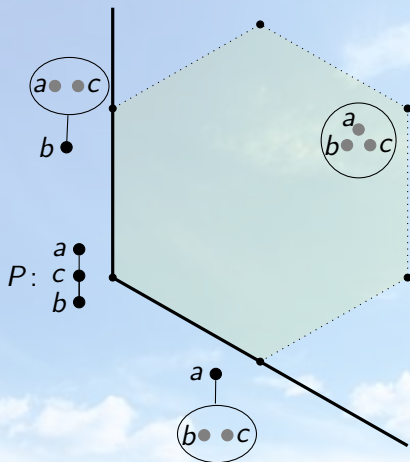


Figure: The cone $\Pi(z^P)$ for P , and the conforming preorders of the four faces of this cone

Modular function

$z : P(I) \rightarrow \mathbb{R} \cup \{\infty\}$ is **modular**:

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- In bijection with pairs (P, α) where $\alpha : \text{bubbles of } P \rightarrow \mathbb{R}$.
- Any z^P is modular,

Hopf algebra

- SM : vector space generated by submodular functions.
- MOD : vector space generated by modular functions

Hopf algebra

- SM : vector space generated by submodular functions.
- MOD : vector space generated by modular functions

Coproduct: $SM \xrightarrow{\Delta} SM \otimes SM$

$$z \xrightarrow{\Delta} \sum_{z(S) < \infty} z|_S \otimes z/S$$

Makes (SM, Δ, μ) a Hopf algebra.

Hopf algebra

- MOD : vector space generated by modular functions

Internal coproduct: $SM \xrightarrow{\delta} SM \otimes MOD$

$$z \mapsto \sum_{P \text{ conforms to } z} z_P \otimes z^P.$$

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- SM becomes a **comodule bialgebra** over MOD.

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- Explanation:
 - Comodules over MOD form a symmetric monoidal category.
 - SM is a bialgebra in this category.

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- Explanation:
 - Comodules over MOD form a symmetric monoidal category.
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- Restricting SM to matroids one gets “cointeraction for matroids”.

Polynomial from cointeraction

- $(\text{MOD}, \Delta, \delta, \mu)$ becomes a double bialgebra. Recent theory of L. Foissy gives a unique double bialgebra morphism $\text{MOD} \rightarrow \mathbb{Q}[x]$.

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- Composing get canonical polynomial:

$$\text{SM} \rightarrow \mathbb{Q}[x].$$

Counts lattice points in the **interior** of all maximal cones of normal fan of $\Pi(z)$.

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Thank you!