Order structures in mathematics

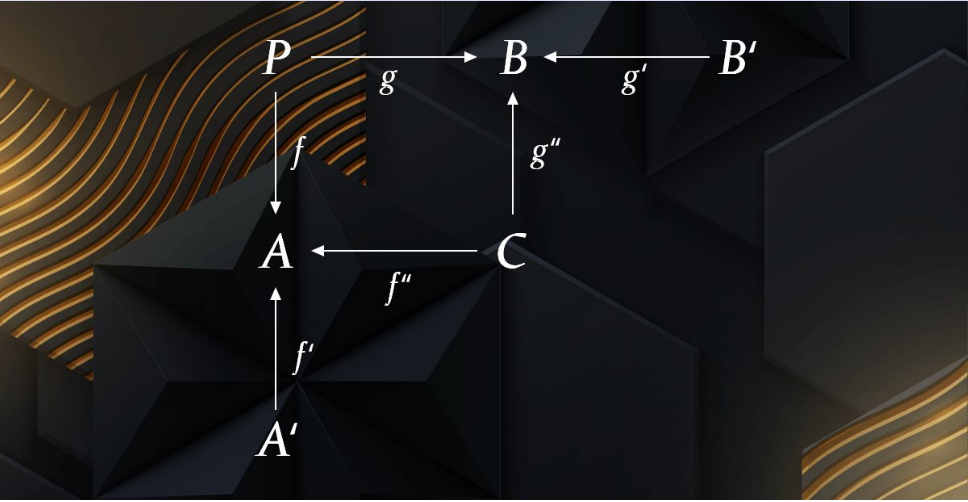
Gunnar Fløystad

September 7, 2024

Highlights

1. Categories, order and symmetry
2. Preorders
3. Topologies
4. Associative algebras
5. Subgroups of matrix groups
6. Ideals in polynomial ring
7. Bipartite graphs
8. Braid cones

Categories and preorders



Categories

Category: Objects and Morphisms

- May compose:

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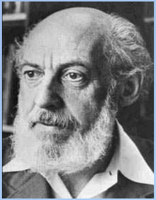
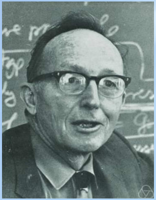
Category theory:

- External:
  - How do mathematical objects relate?
  - "Social life of mathematical objects."
- Internal: Essence of a structure may be a category

Pioneers category theory

S. Eilenberg and S. MacLane

- *General theory of natural equivalences*, Transactions AMS (1945)
- Grothendieck (1957): Abelian categories,
- Kan (1959): Adjoints

Simplest categories

Symmetry

- Category with one object: Monoid
- If also all morphisms are invertible: Group

Simplest categories

Order

Preorder

Category with at most one morphism between any two objects:

Simplest categories

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Preorder: Set V with a relation $\leq$ such that:

- Reflexive: $a \leq a$ for each $a \in V$
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$\mathrm{Pre}(V)$: all preorders on V

Preorders with two elements

$V = \{a, b\}$, two distinct elements

	$a \leq b$	$b \leq a$
1.	NO	NO
2.	YES	NO
3.	NO	YES
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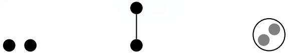
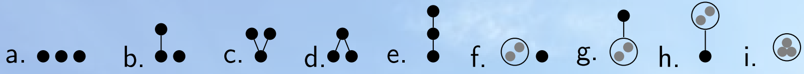
Isomorphism classes:



Preorders with three elements



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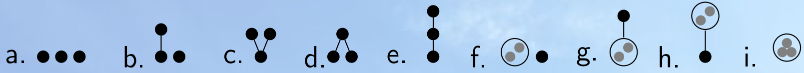

- Discrete preorder: a. Coarse preorder: i.

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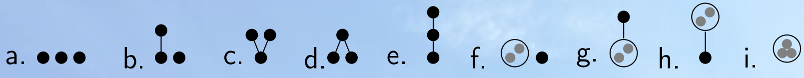

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- Partial orders: a, b, c, d, e.

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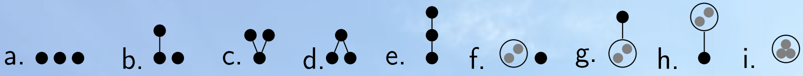

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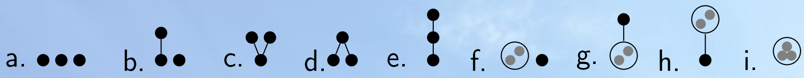

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- Equivalence relations: a, f, i.

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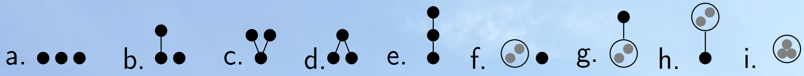

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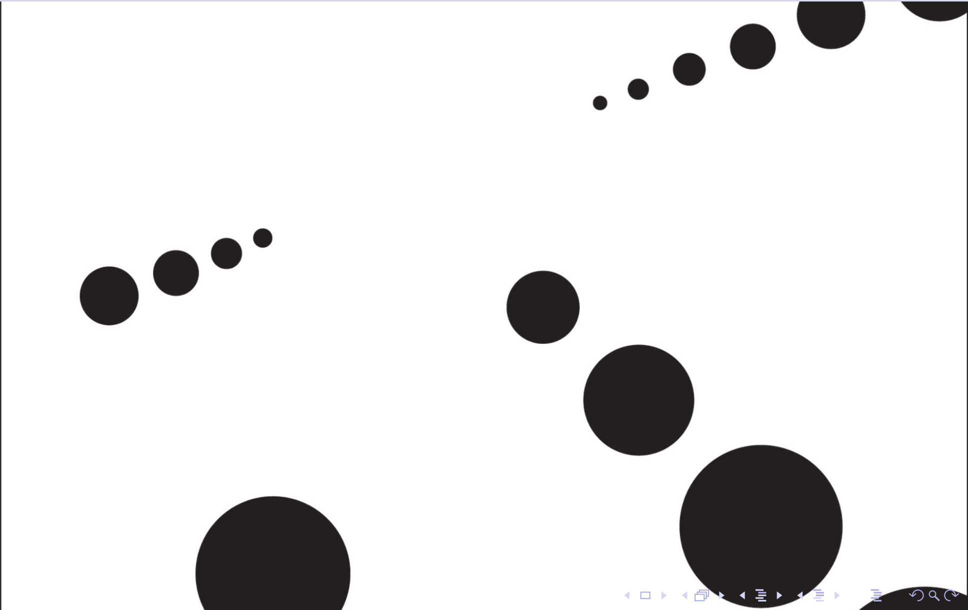
$\mathrm{Pre}(V)$ has 29 elements (when $|V| = 3$).

Finite topologies



Finite topologies

Set X

Topology on X

Family $\mathcal{T}$ of subsets of X:

1. $\emptyset$ and X are in $\mathcal{T}$,
2. **Union:** Let U_i (for $i \in I$) be in $\mathcal{T}$. Then the union $\cup_{i \in I} U_i$ is in $\mathcal{T}$.

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The $U \in \mathcal{T}$ are *open* sets.

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$(X, \leq)$ finite preorder.

Up-set $U \subseteq X$:

$x \in U$ and $x \leq y \;\Rightarrow\; y \in U$.

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Theorem

These are precisely the topologies where X is a finite set.

Pioneers

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Pavel Alexandrov

- 1937: Finite T_0-topologies $\overset{1-1}{\leftrightarrow}$ Partially ordered sets
- 1966: Finite topologies $\overset{1-1}{\leftrightarrow}$ Preorders

Associative algebras

Associative algebras with 1

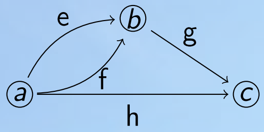
Vector space A with a product $a \cdot b$ fulfilling:

- Associative: $a(bc) = (ab)c$
- Exists 1 such that: $1 \cdot a = a \cdot 1$
- Distributes over addition:

$$a(b+c) = ab + ac, \quad (a+b)c = ab + ac.$$

Directed graph/Quiver

$D = (E, V)$ finite directed graph. Equivalently a map $E \to V \times V$.



Path algebras

Let u and v two vertices. Path a:

$$a : u = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots \xrightarrow{e_n} v_n = v.$$

A: vector space with basis all paths in D.

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- $a : u \to v$ and $b : v \to w$ paths: Product ab is the composed path,
- If end vertex of a is different from start vertex of b: Define $ab := 0$.

A: path algebra of D.

Path algebras with relations

If $r_1, r_2, \ldots, r_p$ are elements of A, may form:

Algebra of relations

$$B = A/(r_1, r_2, \ldots, r_p)$$

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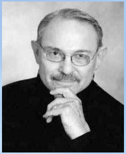
Assume k algebraically closed field (Ex: $k = \mathbb{C}$).

Theorem (60's?)

Any finite dimensional associative algebra over the field k is (up to Morita equivalence) a path algebra with relations.

Pioneers

Kitii Morita



Hyman Bass

- 1958: Morita equivalence
- 1960's: Ideas circulated by Hyman Bass

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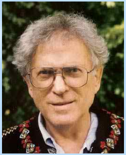
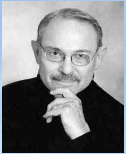


Maurice Auslander



Hyman Bass

Idun Reiten

- 1958: Morita equivalence
- 1960's: Ideas circulated by Hyman Bass

- 1970's and on: Developed theory of Artin algebras

Matrix groups

Matrix

Invertible

Upper triangular matrices form a subgroup:

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

- k field (like $\mathbb{R}$ or $\mathbb{C}$).
- $X = \{x, y, z, \cdots, \}$ finite set, and
- $V = kX$ the vector space with basis X.

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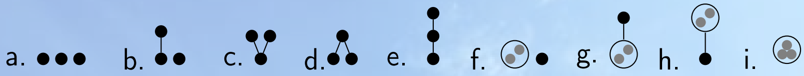
- Upset $U \subseteq X \rightsquigarrow$ subspace $kU \subseteq kX = V$.
- Look at $g : V \to V$ such that $g(kU) \subseteq kU$ for each upset U of X.
- These g make a subgroup of the matrix group.

Each preorder gives a matrix group

Contains the diagonal matrices



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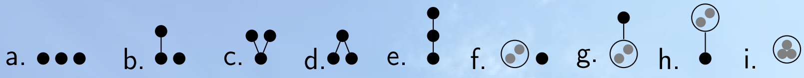

$$\overset{a.}{\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}}, \overset{b.}{\begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}}, \overset{c.}{\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}}, \overset{d.}{\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}}, \overset{e.}{\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}},$$

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Matrix subgroups and preorders

Let $V = kX$. Invertible matrices are invertible linear transformations $g : V \to V$.

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Theorem (Borevich, 81)

Connected subgroups containing the diagonal matrices

$$\updownarrow 1-1$$

Preorders on X

Polynomials

Polynomial rings

and relations

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Ideal $I \subseteq k[x, y]$ is all:

$$p(x, y) \cdot (x^2 + 2xy) + q(x, y) \cdot (y^2 - x - y - 1)$$

where $p(x, y), q(x, y) \in k[x, y]$.

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So $B = k[x, y]/I$.

Degeneration

$$u = x^2 + 2xy, \quad v = y^2 - x - y - 1.$$

Coordinate change $x \to \frac{1}{t^3}x$ and $y \to \frac{1}{t^2}y$. Get:

$$u_1 = \frac{1}{t^6}x^2 + 2\frac{1}{t^5}xy, \quad v_1 = \frac{1}{t^4}y - \frac{1}{t^3}x - \frac{1}{t^2}y - 1.$$

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Let $t \to 0$. Get degeneration:

$$x^2, \quad y^2.$$

Monomial ideal

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There is a map:

$$\text{Monomial ideals in } k[X] \longrightarrow \mathrm{Pre}(X).$$

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Whenever xm is a monomial in I, then ym is a monomial in I.

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Degeneration II

$$x^2, \quad y^2$$

Coordinate change $y \to y + \frac{1}{t}x$:

$$x^2, \quad (y + \frac{1}{t}x)^2 = \frac{1}{t^2}x^2 + 2\frac{1}{t}xy + y^2.$$

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Reorganization $\rightsquigarrow$ relations:

$$x^2, \quad xy + \frac{t}{2}y^2, \quad y^3.$$

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This cannot be further degenerated

The most degenerate ideals

Question

Which ideals are as degenerate as possible?
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Theorem (Folklore, around 1985)

A monomial ideal is as degenerate as possible

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Associated preorder is total.

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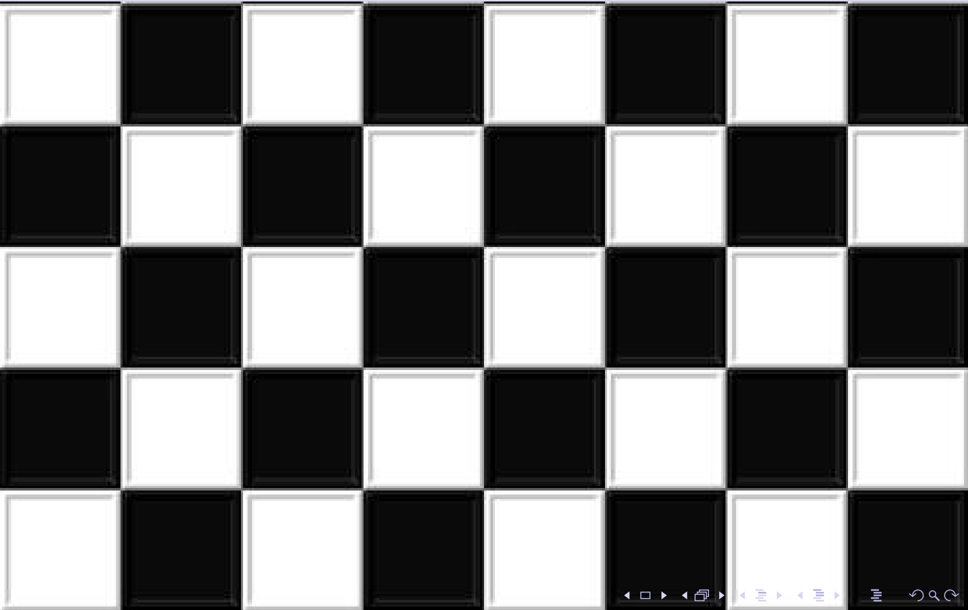
David Bayer

Andre Galligo

Bipartite graphs



Bipartite graphs

Graph with vertices $1, 2, \ldots, n$. Edge

$$\overset{i}{\bullet} - \overset{j}{\bullet} \rightsquigarrow \text{ monomial } x_i x_j.$$

Make these monomials relations in $k[x_1, \ldots, x_n]$.

$$\text{Quotient algebra } B = k[x_1, \ldots, x_n](x_i x_j \mid \{i, j\} \text{ edge}).$$

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Question

When is B a Cohen-Macaulay ring?

(Basic niceness criterion in commutative algebra).

Bipartite graphs

Start with poset P

Make two copies P_1 and P_2. Make bipartite graph:

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Jürgen Herzog and Takayuki Hibi

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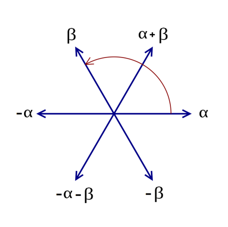

Weyl chambers when $n = 3$.

Braid cones and preorders

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Bijection

Braid cones in $\mathbb{R}^V \overset{1-1}{\longleftrightarrow}$ Preorders on the set V

Order

1. Categories, order and symmetry
2. Preorders
3. Finite topologies
4. Associative algebras
5. Matrix groups
6. Polynomial ideals
7. Bipartite graphs
8. Braid cones


Takk!