

Polarizations of artin monomial ideals define triangulated balls

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Highlights

Polarizations

- 1 Polarizations
- 2 Geometric situation
- 3 Alexander duality
- 4 Cell complexes
- 5 Higher trees

Squarefree monomial ideals

V a finite set (vertices)

Simplicial complex on V :

Family Δ of subsets of V such that: $R \in \Delta$ and $S \subseteq R$ implies $S \in \Delta$.

Squarefree monomial ideals

- V a finite set \rightsquigarrow polynomial ring $k[V] = k[v]_{v \in V}$.
- $S \subseteq V \rightsquigarrow$ squarefree monomial $m(S) = \prod_{s \in S} s$.

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Squarefree monomial ideals $\xleftrightarrow{1-1}$ Simplicial complexes
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$S \in X(I)$ iff monomial $m(S) \notin I$.

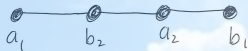
Stanley-Reisner correspondence

(ab, bc, cd)



↓
New names
↓

$(a_2 a_1, a_1 b_1, b_1 b_2)$



Separating artin monomial ideal

$$\begin{aligned} \mathfrak{j} = (a, b)^2 = (a^2, ab, b^2) &\xrightarrow{\text{separate } a} (a_1 a_2, a_1 b, b^2) \\ &\xrightarrow{\text{separate } b} J = (a_1 a_2, a_1 b_1, b_1 b_2) \end{aligned}$$

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$$\frac{k[a_1, a_2, b_1, b_2]}{J} / (b_1 - b_2) = \frac{k[a_1, a_2, b]}{(a_1 a_2, a_1 b, b^2)}.$$

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Well-behaved:

$b_1 - b_2$ not zero divisor in $k[a_1, a_2, b_1, b_2]/J$.

$b_1 - b_2, a_1 - a_2$ is *regular* sequence in $k[a_1, a_2, b_1, b_2]/I$.

Separating artin monomial ideal

$$j = (a, b)^2 = (a^2, ab, b^2) \xrightarrow{\text{separate } a} (a_1 a_2, a_1 b, b^2) \\ \xrightarrow{\text{separate } b} J = (a_1 a_2, a_1 b_1, b_1 b_2)$$

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- J is a **polarization** of j .

Polarization

- Surjection $A \cup B \cup \dots \cup E \xrightarrow{P} \{a, b, \dots, e\}$
(like $\{a_1, a_2, b_1, b_2\} \rightarrow \{a, b\}$).

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J is a **polarization** of \mathfrak{J} :

Quotienting $k[A, B, \dots, E]/J$ with a **regular** sequence of **variable differences** gives $k[a, b, \dots, e]/\mathfrak{J}$.

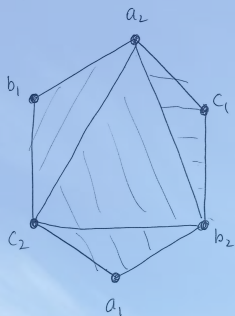
Examples

Two distinct polarizations of $(a, b, c)^2$ (up to isomorphism):

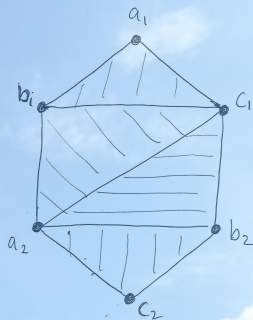
$$J_1 = (a_1 a_2, a_1 b_1, b_1 b_2, a_1 c_1, b_1 c_1, c_1 c_2),$$

$$J_2 = (a_1 a_2, a_1 b_2, b_1 b_2, a_1 c_2, b_1 c_2, c_1 c_2)$$

Triangulations of hexagons



$$J_1 = (a_1 a_2, a_1 b_2, b_1 b_2, a_1 c_1, b_1 c_1, c_1 c_2)$$



$$J_2 = (a_1 a_2, a_1 b_2, b_1 b_2, a_1 c_2, b_1 c_2, c_1 c_2)$$

Conjecture

Almousa, Fløystad, Lohne (2019/2022)

Let J be polarization of an [artin](#) monomial ideal.
The simplicial complex associated to J
is a [triangulated ball](#).

Now 2024

Theorem

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Polarizations

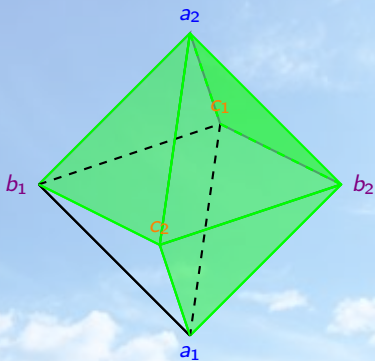
$$J_1 = (a_1 a_2, a_1 b_1, b_1 b_2, a_1 c_1, b_1 c_1, c_1 c_2) \subseteq C = (a_1 a_2, b_1 b_2, c_1 c_2)$$

Polarizations

$$J_1 = (a_1 a_2, a_1 b_1, b_1 b_2, a_1 c_1, b_1 c_1, c_1 c_2) \subseteq C = (a_1 a_2, b_1 b_2, c_1 c_2)$$

- C is a complete intersection,
- Simplicial complex $X(C)$ is an octahedron,
- $X(J_1)$ is a full-dimensional subcomplex of $X(C)$.

Polarization embedded on octahedron



Boundaries of simplices

Monomial $m(C) \rightsquigarrow$
boundary $\partial\Delta(C)$

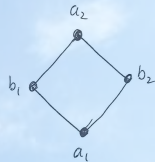
$$m_2 = (a_1 a_2) \longleftrightarrow \begin{array}{cc} 1 & 2 \\ \circ & \circ \end{array}$$

$$m_3 = (a_1 a_2 a_3) \longleftrightarrow \begin{array}{c} 2 \\ \circ \\ \diagup \quad \diagdown \\ \circ \quad \quad \circ \\ 1 \quad \quad 3 \end{array}$$

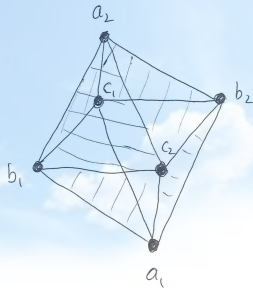
$$m_4 = (a_1 a_2 a_3 a_4) \longleftrightarrow \begin{array}{c} 4 \\ \circ \\ \diagup \quad \quad \diagdown \\ \circ \quad \quad \quad \circ \\ \diagdown \quad \diagup \\ \circ \quad \quad \circ \\ 1 \quad \quad 2 \quad \quad 3 \end{array}$$

Joins of boundaries

$$C_2 = (a_1 a_2, b_1 b_2)$$



$$C_3 = (a_1 a_2, b_1 b_2, c_1 c_2)$$



Joins of boundaries

- Complete intersection of monomials

$$C = (m(A), m(B), \dots, m(E)).$$

Joins of boundaries

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$$X(C) = \partial\Delta(A) * \partial\Delta(B) * \dots * \partial\Delta(E).$$

Joins of boundaries

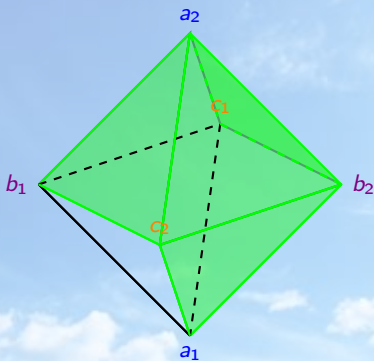
- Complete intersection of monomials
 $C = (m(A), m(B), \dots, m(E))$.

- Associated simplicial complex is join:

$$X(C) = \partial\Delta(A) * \partial\Delta(B) * \dots * \partial\Delta(E).$$

- When: J is a polarization of an artin $j \subseteq k[a, b, \dots, e]$,
- Then: $X(J)$ is full-dimensional CM subcomplex of the join complex $X(C)$.

Polarization embedded on octahedron



Alexander dual

J : squarefree monomial ideal

Alexander dual ideal $I = J^\vee$:

Generated by the monomials (in the polynomial ring) having (non-trivial) **common divisor** with **every** monomial in J .

Alexander dual

$$J_1 = (a_1 a_2, a_1 b_1, b_1 b_2, a_1 c_1, b_1 c_1, c_1 c_2),$$

$$J_2 = (a_1 a_2, a_1 b_2, b_1 b_2, a_1 c_2, b_1 c_2, c_1 c_2)$$

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Alexander dual ideals:

$$I_1 = J_1^\vee = (a_1 b_1 c_1, a_1 b_1 c_2, a_1 b_2 c_1, a_1 b_1 c_1),$$

$$I_2 = J_2^\vee = (a_1 b_1 c_1, a_1 b_1 c_2, a_1 b_2 c_2, a_2 b_2 c_2)$$

Alexander dual

Each tuple $u = (a_i, b_j, \dots, e_\ell) \in A \times B \times \dots \times E$ gives a **rainbow monomial**

$$m(u) = a_i b_j \dots e_\ell \in k[A, B, \dots, E].$$

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$$m(u) = a_i b_j \dots e_\ell \in k[A, B, \dots, E].$$

Any subset $T \subseteq A \times B \times \dots \times E$ induces a **rainbow** monomial ideal

$$I(T) = (m(u) \mid u \in T).$$

Alexander dual is rainbow ideal

Proposition

When: J is a polarization, the Alexander dual ideal $I = J^\vee$ is a
rainbow ideal

$$I = I(T).$$

Alexander dual is rainbow ideal

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Since J is Cohen-Macaulay, $I(T)$ has **linear resolution**.

Alexander dual is rainbow ideal

Proposition

When: J is a polarization, the Alexander dual ideal $I = J^\vee$ is a **rainbow ideal**

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Since J is Cohen-Macaulay, $I(T)$ has **linear resolution**.

Conversely, if the rainbow ideal $I(T)$ has linear resolution, its Alexander dual $J = I(T)^\vee$ is a **polarization**.

Linear resolution

Theorem

When $I(T)$ has linear resolution, it is a **constructible** ideal.

Consequence: $X(J)$ is a **constructible simplicial complex**.

Linear resolution

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Theorem of Björner

When: X is a simplicial complex which is:

- Constructible,
- Every codimension one face is on at most two facets (easy to see for $X(J)$),

Then: X is a **triangulation of a ball or a sphere**.

Now 2024

Fundamental theorem on geometry of monomial ideals

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(Save when j is a complete intersection: Then J defines a sphere.)

Linear resolution

We shift perspective to T

For $T \subseteq A \times B \times \cdots \times E$:

- When does $I(T)$ have linear resolution?
- What does such T “look like”?

Join and product polytopes are dual

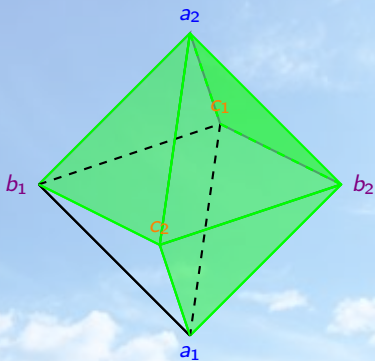
The dual (polar) of the join polytope:

$$\partial\Delta(A) * \partial\Delta(B) * \cdots * \partial\Delta(E)$$

is the product polytope:

$$\Delta(A) \times \Delta(B) \times \cdots \times \Delta(E).$$

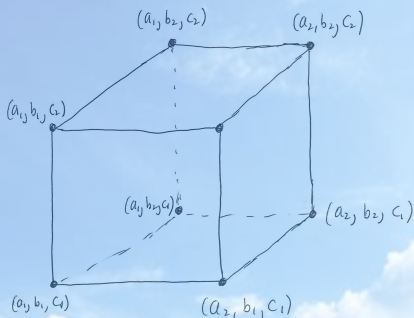
The cube is dual to the octahedron



Vertices on the cube

Cube

$$\Delta(A) \times \Delta(B) \times \Delta(C)$$



$$\Delta(\{a_1, a_2\}) = \begin{array}{c} a_1 \quad a_2 \\ \bullet \text{---} \bullet \end{array}$$

$$\Delta(\{b_1, b_2\}) = \begin{array}{c} b_1 \quad b_2 \\ \bullet \text{---} \bullet \end{array}$$

$$\Delta(\{c_1, c_2\}) = \begin{array}{c} c_1 \quad c_2 \\ \bullet \text{---} \bullet \end{array}$$

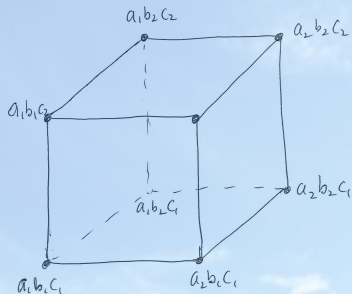
Vertices on the product

The set $A \times B \times \cdots \times E$ correspond to the **vertices** of the **product polytope**:

$$\Pi(A, B, \cdots, E) = \Delta(A) \times \Delta(B) \times \cdots \times \Delta(E).$$

Vertices on the cube

Labellings



Faces on the product

The product polytope $\Pi(A, B, \dots, E)$ is a cell complex.
The cells are the products:

$$\Delta(A_0) \times \Delta(B_0) \times \cdots \times \Delta(E_0) \subseteq \Pi(A, B, \dots, E)$$

for non-empty subsets:

$$A_0 \subseteq A, \dots, E_0 \subseteq E.$$

Recall

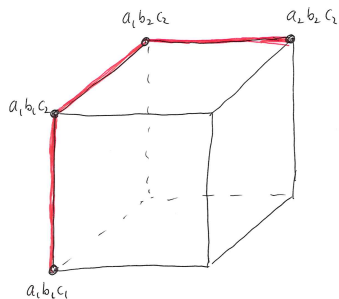
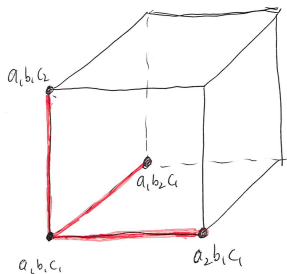
Alexander dual ideals:

$$I(T_1) = J_1^\vee = (a_1 b_1 c_1, a_1 b_1 c_2, a_1 b_2 c_1, a_1 b_1 c_1),$$

$$I(T_2) = J_2^\vee = (a_1 b_1 c_1, a_1 b_1 c_2, a_1 b_2 c_2, a_2 b_2 c_2)$$

Induced subcomplexes on cube

$T \subseteq A \times B \times \cdots \times E$ induces a subcomplex $C(T)$ of the product polytope $\Pi(A, B, \dots, E)$.



Induced subcomplexes on joins

- Cells of $C(T)$: The products $\Delta(A_0) \times \Delta(B_0) \times \cdots \times \Delta(E_0)$ where the product

$A_0 \times B_0 \times \cdots \times E_0$ is subset of T .

Induced subcomplexes on joins

- Cells of $C(T)$: The products $\Delta(A_0) \times \Delta(B_0) \times \cdots \times \Delta(E_0)$ where the product

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- $C(T)$ is a **labelled** (by monomials) cell complex,

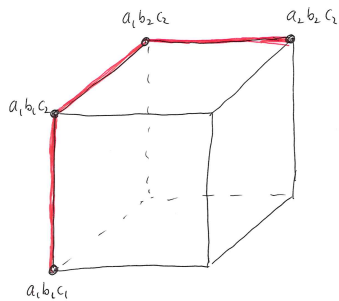
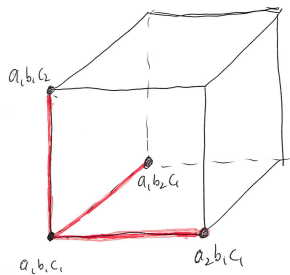
Induced subcomplexes on joins

- Cells of $C(T)$: The products $\Delta(A_0) \times \Delta(B_0) \times \cdots \times \Delta(E_0)$ where the product

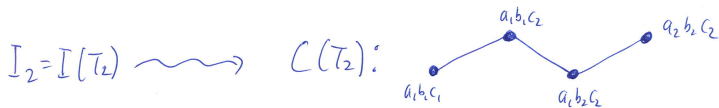
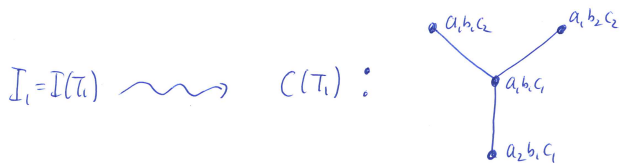
$A_0 \times B_0 \times \cdots \times E_0$ is subset of T .

- $C(T)$ is a **labelled** (by monomials) cell complex,
- $C(T)$ induces a **cellular complex**, $\mathcal{C}_\bullet(T)$.

Labelled cell complex



Labelled cell complex



Cellular complex

$$S = k[a_1, a_2, b_1, b_2, c_1, c_2]$$

$$\mathcal{C}_\bullet(T_i) : S(-3)^4 \longleftarrow S(-4)^3.$$

- 4 vertices
- 3 edges

Cellular resolution

Theorem

When $I(T)$ has linear resolution,
 $\mathcal{C}_\bullet(T)$ is its **minimal free resolution**.

Cellular resolution

Theorem

When $I(T)$ has linear resolution,
 $\mathcal{C}_\bullet(T)$ is its **minimal free resolution**.

Question:

What does the cell complex $\mathcal{C}(T)$ look like when $I(T)$ has linear resolution?

Acyclic T

For a product $U = A_0 \times B_0 \times \cdots \times E_0$ of subsets of A, B, \dots, E ,
the **restriction**:

$$T|_U = T \cap U.$$

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$I(T)$ has linear resolution



The reduced homology $\tilde{H}_i(C(T|_U))$ **vanishes** for **all** U and **all** $i \geq 0$.

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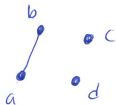
The reduced homology $\tilde{H}_i(C(T|_U))$ **vanishes** for **all** U and **all** $i \geq 0$.

- Such T are **acyclic**.
- $C(T)$ is a “**higher-dimensional tree**”.

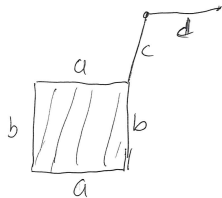
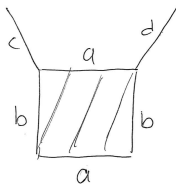
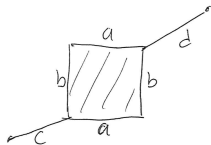
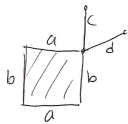
Polarizations I

- $j = (a^2, b^2, c^2, d^2) + (ac, ad, bc, bd, cd)$
 $= j_{\square} + j_{\Delta}$

- j_{Δ} is Stanley-Reisner ideal of :



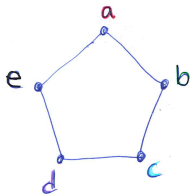
Cell complexes $C(T)$ corresponding to polarizations of j :



Polarizations II

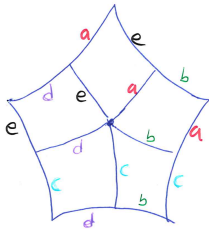
$$\begin{aligned} \bullet \quad \mathcal{J} &= (a^2, b^2, c^2, d^2, e^2) + (ac, bd, ce, da, eb) \\ &= \mathcal{J}_\square + \mathcal{J}_\Delta \end{aligned}$$

$\bullet \quad \mathcal{J}_\Delta$ is Stanley-Reisner ideal of:



Only standard polarization

Cell complex $\mathcal{C}(\mathcal{J})$ is:



Thank you!