Eight times four bialgebras of hypergraphs, cointeractions, and chromatic polynomials

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Highlights

Hypergraphs

- E and V finite sets.
- $P(V)$ power set of V (all subsets of V).

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No restrictions

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- One edge, one vertex:

$h :$  $,\quad \chi$  $(x) = 0.$

Double bialgebras

- Double bialgebra (B, μ, Δ, δ) (notion L. Foissy 2022):
  - Commutative multiplication μ,
  - Two coproducts Δ, δ which cointeract,
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- Eight double bialgebra structures on H,
- Hypergraph $h \rightsquigarrow$ eight polynomials

Quartets of bialgebras

- We produce eight quartets of bialgebras.
- Five are genuinely distinct. (The other three arise as from the opposite coproduct Δ^{op} when Δ is not cocommutative.)
- Four of these five quartets, give eight double bialgebras.

Coproducts for hypergraphs

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Example

Counit ϵ_Δ sends $(E, \emptyset) \mapsto 1$, otherwise to 0.

Bialgebras

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Bialgebras

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H vector space with basis (iso-classes of) hypergraphs.

- Bialgebra $(H, \mu, \Delta, \eta, \epsilon)$ graded by $|V|$,

Contraction

- Hypergraph $F \xrightarrow{f} P(V)$.
- Consider surjections $V \to W$ such that:
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- Consider surjections $V \to W$ such that:
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- V/F: Largest possible W. The connected components of the hypergraph f.

Extraction/contraction coproduct

$E^* \subseteq E$ edges whose vertex set is non-empty.

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where complement $F^c = E \backslash F$.

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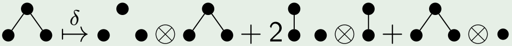
Counit:

$$\epsilon_\delta(h) = \begin{cases} 1, & h = (\emptyset, V) \\ 0, & \text{otherwise} \end{cases}$$

Examples extraction/contraction

Example



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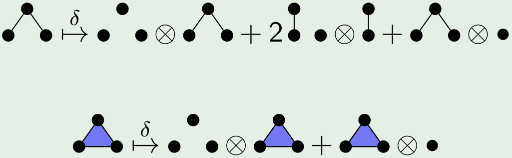

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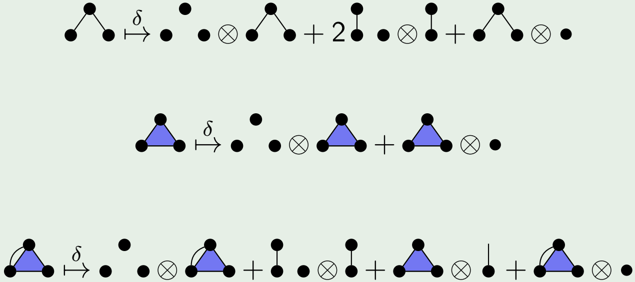

Two bialgebras

- $(H, \mu, \Delta, \eta, \epsilon_\Delta)$
  (This is even a Hopf algebra.)
- $(H, \mu, \delta, \eta, \epsilon_\delta)$

Note: $\eta(1) = (\emptyset, \emptyset)$.

Cointeracting bialgebras

Bialgebras:

- $\mathcal{H} = (H, \mu_{\mathcal{H}}, \Delta_{\mathcal{H}}, \eta_{\mathcal{H}}, \epsilon_{\mathcal{H}})$
- $\mathcal{B} = (B, \mu_{\mathcal{B}}, \Delta_{\mathcal{B}}, \eta_{\mathcal{B}}, \epsilon_{\mathcal{B}})$

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$\mathcal{H}$ is a comodule bialgebra over $\mathcal{B}$:

1. $\mathcal{H}$ is left comodule over $\mathcal{B}$: $\mathcal{H} \xrightarrow{\Phi} \mathcal{B} \otimes \mathcal{H}$,
2. $\Delta_{\mathcal{H}}$ and $\epsilon_{\mathcal{H}}$ are morphism of left comodules,
3. Φ is a unital algebra morphism.

Double bialgebras, L.Foissy 2022

Special case of cointeracting bialgebras

(B, μ, η) algebra: $\mu : B \otimes B \to B, \quad \eta : k \to B,$

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- (B, μ, Δ) is comodule bialgebra over (B, μ, δ) with $\Phi = \delta$.

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Axiom to be checked:

$$\begin{array}{ccc} B & \xrightarrow{\delta} & B \otimes B \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ B \otimes B & & B \otimes B \otimes B \\ \downarrow{\scriptstyle \delta\otimes\delta} & & \uparrow{\scriptstyle \mu\otimes 1\otimes 1} \\ B \otimes B \otimes B \otimes B & \xrightarrow{\tau_{23}} & B \otimes B \otimes B \otimes B \end{array}$$

commutes

Double bialgebra

Example

Polynomial ring $\mathbb{Q}[x]$ with:

Double bialgebra

Example

Polynomial ring $\mathbb{Q}[x]$ with:

- $x \overset{\Delta}{\mapsto} x \otimes 1 + 1 \otimes x$,
- $x \overset{\delta}{\mapsto} x \otimes x$.

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- $x \overset{\delta}{\mapsto} x \otimes x$.

$$\mathbb{Q}[x] \otimes \mathbb{Q}[x] \cong \mathbb{Q}[x, y], \quad x \otimes 1 \mapsto x, \quad 1 \otimes x \mapsto y$$

Double bialgebra

Example

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- $x \overset{\delta}{\mapsto} x \otimes x$.

$$\mathbb{Q}[x] \otimes \mathbb{Q}[x] \cong \mathbb{Q}[x, y], \quad x \otimes 1 \mapsto x, \quad 1 \otimes x \mapsto y$$

- $x \overset{\Delta}{\mapsto} x + y$,
- $x \overset{\delta}{\mapsto} xy$.

Double bialgebras

Hypergraphs without empty edges $\rightsquigarrow H^\circ \subseteq H$.

- (H, μ, Δ) becomes a comodule bialgebra over (H°, μ, δ)
- $(H^\circ, \mu, \Delta, \delta)$ is a double bialgebra

Double bialgebras and polynomials

L.Foissy 2022

- (B, μ, Δ, δ) is a double bialgebra.

Theorem

If (B, μ, Δ) is a Hopf algebra, there is a unique double bialgebra morphism:

$$\chi : (B, \mu, \Delta, \delta) \to (\mathbb{Q}[x], \Delta, \delta).$$

Graph bialgebras

D double bialgebra of graphs (Schmitt 1994)
$\rightsquigarrow$ unique $\chi : D \to \mathbb{Q}[x]$.

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Theorem (Foissy 2021)

$\chi_G(x) =$ *chromatic polynomial of* G.

Graph bialgebras

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Theorem (Foissy 2021)

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Example

$G =$  , $\quad \chi_G(x) = x(x-1) = x^2 - x$

Examples

$(H^{\circ}, \mu, \Delta, \delta)$ double bialgebra of hypergraphs $\rightsquigarrow$ unique

$$\phi : H^{\circ} \to \mathbb{Q}[x].$$

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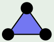
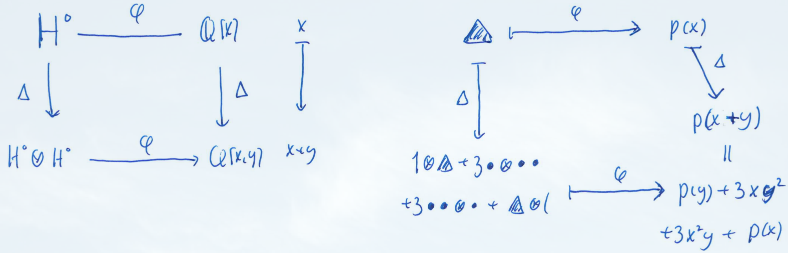
Example

$h =$  , $\quad \phi_h(x) = x^3 - x$

ϕ respects Δ

One edge with three vertices



Equation $p(x+y) = p(y) + p(x) + 3xy^2 + 3x^2y$

$p(x) = x^3 + ax$

ϕ respects δ

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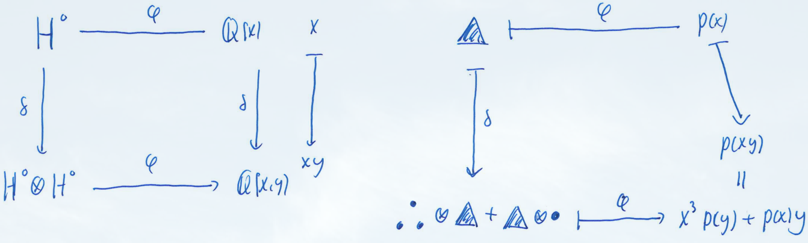

Equation $p(xy) = x^3 p(y) + p(x)\, y$

$$\Downarrow$$

$$p(x) = b(x^3 - x)$$

Chromatic polynomial: one edge with three vertices

$$p(x) = x^3 + ax = b(x^3 - x).$$

$$p(x) = x^3 - x$$

More examples

- V has n vertices
- $h = (\emptyset, V)$, no edges: $\chi_h(x) = x^n$

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- V has n vertices
- $h = (\emptyset, V)$, no edges: $\chi_h(x) = x^n$
- $h = (e, V)$ single edge with all n vertices: $\chi_h(x) = x^n - x$

Hypergraph chromatic polynomial

History

- $\chi_h(k)$ counts colorings with k vertices such that no edge is monochromatic.
- Chromatic number for hypergraphs: Erdös, Hanal, 1966
- Chromatic polynomial hypergraphs: Helgasson, 1974

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- One edge, one vertex:

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Key idea I

$\mathcal{C}, \mathcal{D}$ categories

<u>Profunctor</u> $F: \mathcal{C} \nrightarrow \mathcal{D}$ is a functor

$$F: \mathcal{C} \longrightarrow \mathrm{Hom}(\mathcal{D}^{op}, \underline{set})$$

Equivalently a functor

$$F: \mathcal{C} \times \mathcal{D}^{op} \longrightarrow \underline{set}$$

Categories enriched in $\{0<1\}$, meaning:

$\mathrm{Hom}(A,B)$ is an object of $\{0<1\}$

$\mathrm{Hom}(A,B)=0 \longleftrightarrow \mathrm{Hom}(A,B)=\emptyset$

$\mathrm{Hom}(A,B)=1 \longleftrightarrow \mathrm{Hom}(A,B)$ one element.

Such a category is a <u>preorder</u>

In particular <u>posets</u> are such categories

P, Q posets

<u>Profunctor</u> $f: P \nrightarrow Q$ is an order preserving map:

$$f: P \longrightarrow \mathrm{Hom}(Q^{op}, \{0<1\})$$

Equivalently an o-p map:

$$f: P \times Q^{op} \longrightarrow \{0<1\}$$

G. Fløystad: Profunctors between posets and Alexander duality, Applied Category Theory (2023)

Note: $P \times Q^{op} \to \{0<1\} \rightsquigarrow Q^{op} \xrightarrow{g} \mathrm{Hom}(P, \{0<1\})$

<u>Dual</u>: $g: Q^{op} \nrightarrow P^{op}$

<u>Hypergraphs</u> is a profunctor $h: E \nrightarrow V$

Immediately get <u>dual</u> hypergraph

$$h^d: V \nrightarrow E$$

Observation II

Coloring of graph: Map $V \xrightarrow{c} C$ subject to conditions.

Coloring of (E, V) by $C = X \cup Y$

$\Updownarrow$

Choosing $U \subseteq V$ and color:

1) $(E|_U, U)$ with X
2) $(E|_{U^c}, U^c)$ with Y

Coloring (E, V) by $C = X \times Y$

$\Updownarrow$

Choose $F \subseteq E^*$ and color:

1) (F, V) by X
2) $(F^c, V/F)$ by Y

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- Hypergraph identifies as relation between E and V,
- Get dual hypergraph:

$$h^d : V \to P(E).$$

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$$P(V) \xrightarrow{c} P(V)$$
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$\rightsquigarrow$ complementary hypergraph $E \xrightarrow{c \circ h} P(V)$.

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Get four hypergraphs:

$$E \xrightarrow{h} P(V), \quad E \xrightarrow{c \circ h} P(V), \quad V \xrightarrow{h^d} P(E), \quad V \xrightarrow{(c \circ h)^d} P(E).$$

Hypergraphs as relations

$$E \overset{V}{\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix}}, \quad E \overset{V}{\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}} \text{ complement}$$

$$\text{dual } E \overset{V}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{vmatrix}}, \quad E \overset{V}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{vmatrix}} \text{ complement dual}$$

Second coproduct

Use Δ on dual hypergraph and translate back

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$$(E, V) \mapsto \sum_{F \subseteq E} (F, V_{|F}) \otimes (F^c, V_{|F^c}),$$

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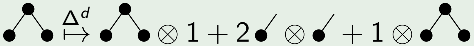
Example



Dual double bialgebra

- Duality on hypergraphs, gives coproducts Δ^d and δ^d
- The dual double bialgebra $(H^d, \mu, \Delta^d, \delta^d)$
  $\rightsquigarrow$ chromatic hypergraph polynomial $\chi_h^d(x)$

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- $\chi_h^d(x)$ counts edge colorings of H such that for each vertex v: edges containing v not all of same color.

Cointeracting bialgebras

Complementing is involution on hypergraphs
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Composite involution $c \circ d$ $\rightsquigarrow$ coproducts Δ^{cd} and δ^{cd}
$\rightsquigarrow$ double bialgebra $(H^{cd}, \mu^c, \Delta^{cd}, \delta^{cd})$.

Four double bialgebras on hypergraphs

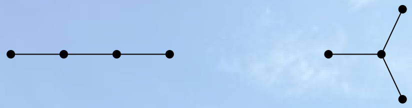
Four chromatic polynomials

From $H^{\circ}, H^{d}, H^{c}, H^{cd}$

$$\chi_h(x), \quad \chi_h^{d}(x), \quad \chi_h^{c}(x), \quad \chi_h^{cd}(x).$$

Trees

Trees with three edges: path and star:



Chromatic polynomials of path:

$$\chi(x) = x(x-1)^3, \quad \chi^d(x) = 0, \quad \chi^c(x) = x(x-1)^3, \quad \chi^{cd}(x) = 0.$$

Chromatic polynomials of star:

$$\chi(x) = x(x-1)^3, \quad \chi^d(x) = 0, \quad \chi^c(x) = x^2(x-1)(x-2),$$

$$\chi^{cd}(x) = x(x-1)(x-2).$$

Key idea III

$f : R \longrightarrow S$ map of sets

WRITE: Powerset $P(R) = \mathrm{Hom}(R, \{0<1\})$

Induces: $f^* : P(S) \longrightarrow P(R)$

$U \subseteq S \longmapsto f^{-1}(U) \subseteq R$

Left adjoint: $f^! : P(R) \longrightarrow P(S)$

$U \longmapsto f(U)$

Right adjoint: $f^{i} : P(R) \longrightarrow P(S)$

$U \longmapsto \left(f(U^c)\right)^c$

Hypergraph $h : E \longrightarrow P(V)$

Subset $U \overset{i}{\subseteq} V$

① Pull-back

$$\begin{array}{ccc} E_{|U} & \longrightarrow & P(U) \\ \downarrow & \ulcorner & \downarrow i^! \\ E & \longrightarrow & P(V) \end{array}$$

② $E \longrightarrow P(V) \xrightarrow{i^*} P(U)$

Restriction descent bialgebra

For graphs: Aguiar, Ardilla, 2017.

$$(E, V) \overset{\Delta'}{\mapsto} \sum_{U \subseteq V} (E_{|U}, U) \otimes (E, U^c).$$

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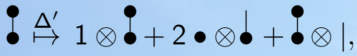
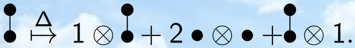


while the coproduct Δ:



Four new bialgebras of hypergraphs

Restriction-descent algebras

$$(H, \mu, \Delta'), \quad (H, \mu, \Delta'^{d}), \quad (H, \mu, \Delta'^{c}), \quad (H, \mu, \Delta'^{cd}).$$

Descent-descent bialgebra

$$(E, V) \overset{\Delta''}{\mapsto} \sum_{U \subseteq V} (E, U) \otimes (E, U^c).$$

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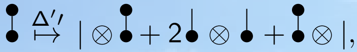

New double bialgebra

L.Foissy 2023

- (H, μ, Δ'') *has* a cointeracting bialgebra (H, μ, δ''),
- Get double bialgebra $(H^\circ, \mu, \Delta'', \delta'')$.

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What is the polynomial $\chi''_h(x)$?

The rainbow polynomial

Single edge with three vertices: $h =$ 

Chromatic polynomials

$$\chi_h(x) = x^3 - x, \quad \chi''_h(x) = x(x-1)(x-2).$$

The rainbow polynomial

Single edge with three vertices: $h =$ 

Chromatic polynomials

$$\chi_h(x) = x^3 - x, \quad \chi''_h(x) = x(x-1)(x-2).$$

- $\chi''_h(k)$ counts colorings where every edge is rainbow: all vertices have different colors,

The rainbow polynomial

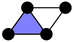
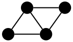
Single edge with three vertices: $h =$ 

Chromatic polynomials

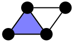
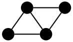
$$\chi_h(x) = x^3 - x, \quad \chi''_h(x) = x(x-1)(x-2).$$

- $\chi''_h(k)$ counts colorings where every edge is rainbow: all vertices have different colors,
- $\chi_h(k)$ counts colorings where every edge *not* monochromatic

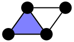
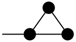
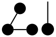
Rainbow polynomials

 and  have same χ''-chromatic polynomial $x(x-1)(x-2)^2$.

Rainbow polynomials

 and  have same χ''-chromatic polynomial $x(x-1)(x-2)^2$.

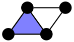
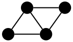
The first gives the quartet:

h : , h^d : , h^c : , h^{cd} : ,

with χ''-polynomials:

$$x(x-1)(x-2)^2, \quad x(x-1)(x-2), \quad x^2(x-1)^2, \quad x^2(x-1).$$

Rainbow polynomials

 and  have same χ''-chromatic polynomial $x(x-1)(x-2)^2$.

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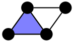
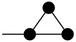
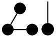
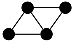
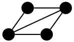
h : , h^d : , h^c : , h^{cd} : ,

with χ''-polynomials:

$$x(x-1)(x-2)^2, \quad x(x-1)(x-2), \quad x^2(x-1)^2, \quad x^2(x-1).$$

The second gives the quartet:

g : , g^d : , g^c : , g^{cd} : .

with χ''-polynomials:

$$x(x-1)(x-2)^2, \quad x(x-1)(x-2)(x^2-5x+7), \quad x(x-1)(x-2)^2$$
$$x(x-1)(x-2)(x^2-5x+7).$$

Thank you!