Combinatorial Hopf algebras from preorder cuts

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Highlights

Part I

1. Examples of Hopf products μ and coproducts Δ

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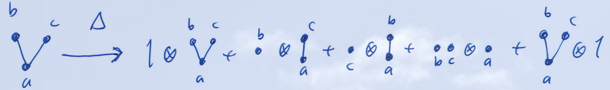
Part II

- The new way: Examples of Hopf algebras

Connes-Kreimer Hopf algebra

Coproduct :



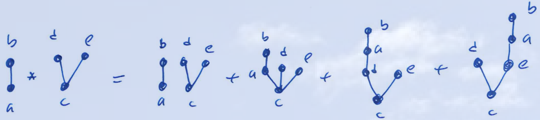
Product :

$$t_1 \times t_2 = t_1 t_2$$

$$t_1 t_2 \times t_3 t_4 t_5 = t_1 t_2 t_3 t_4 t_5$$

Grossmann-Larson Hopf algebra

Product:



Coproduct:

$$t_1 t_2 t_3 \xrightarrow{\Delta} 1 \otimes t_1 t_2 t_3 + t_1 \otimes t_2 t_3 + t_2 \otimes t_1 t_3 + t_3 \otimes t_1 t_2$$
$$+ t_1 t_2 \otimes t_3 + t_1 t_3 \otimes t_2 + t_2 t_3 \otimes t_1 + t_1 t_2 t_3 \otimes 1$$

Malvenuto-Reutenauer Hopf algebra

- S_n: permutations of $1,2,\ldots,n$. $S_0 = \{i\}$
- MR-algebra has basis $\bigcup_{n\geq 0} S_n$,

Coproduct: $3124 \xrightarrow{\Delta}$ "$i \otimes 3124 + 3 \otimes 124 + 31 \otimes 24 + 312 \otimes 4 + 3124 \otimes i$"

$= i \otimes 3124 + 1 \otimes 123 + 21 \otimes 12 + 312 \otimes 1 + 3124 \otimes i$

Product: 12×312 = shuffles of 12 and 534

$= 12534 + 15234 + 15324 + 15342 + 51234$

$+ 51324 + 51342 + 53124 + 53142 + 53412$

Species

set$^\times$: Category of finite sets with bijections

Species S in a category $\mathcal{C}$ is a functor:

$$\begin{aligned} \mathsf{S} : \mathbf{set}^\times &\longrightarrow \mathcal{C} \\ X &\mapsto \mathsf{S}[X] \end{aligned}$$

Usually $\mathcal{C}$ is **set** or **vect**

Example (Example of species S)

$X = \{a, b, c\}$ and $\mathcal{C} = \mathbf{set}$

$$\mathsf{S}[X] = \text{ roooted forests whose vertices are the elements of} X.$$

Monoids and comonoids in species

Partition $X = A \sqcup B$

$$1[X] = \begin{cases} 0/\emptyset, & X \neq \emptyset \\ \{*\}, & X = \emptyset \end{cases}$$

Monoids in species have products

$$S[A] \times S[B] \xrightarrow{\mu_{A,B}} S[X].$$

and unit

$$\eta : 1[X] \to S[X].$$

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Comonoid problem : Works when $\mathcal{C}$ is **vect** but not **set**.

Observations

Hopf monoid H $= (\Delta, \epsilon, \mu, \iota)$ with $H[\emptyset] = \{*\}$.

From examples:

1. Coproduct Δ "better" than μ: It's simpler with Δ.
2. Have natural bases: So more natural with **set** than **vect**

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  - Δ has domain $\mathrm{H}[X]$, while $\Delta' = \mu^*$ has domain $\mathrm{H}[X]^*$.
- Solution: $\mathrm{H}[X]$ comes with basis so canonically $\mathrm{H}[X]^* \cong \mathrm{H}[X]$.

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   Requires two preorders on our objects/elements

How replace vector space with set

Category $\mathbf{set}^m$:

- Objects: sets
- Morphism $X \xrightarrow{f} Y$ is a map

$$\begin{aligned} X &\to \mathrm{Hom}(Y, \mathbb{N}) \\ x &\mapsto \text{multisubset of } Y \end{aligned}$$

Get dual $Y \xrightarrow{Df} X$.

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Now species

$$S : \mathbf{set}^{\times} \to \mathbf{set}^{m}.$$

Diagram of partial maps

$X \xrightarrow{f} Y$ is a partial map if for each $x \in X$

$$f(x) = \{y\} \text{ or } \mathbf{0}(= \emptyset)$$

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$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ Z & \xrightarrow{\delta} & W \end{array} \tag{1}$$

- $Y' = \{y \in Y \mid \gamma(y) \neq \mathbf{0}\}$
- $Z' = \{z \in Z \mid \delta(z) \neq \mathbf{0}\}$
- $X'' = \{x \in X \mid \alpha(x), \beta(x) \neq \mathbf{0}\}$

Definition

This is a partial pull-back diagram (ppbd) if:

Partial pullback diagram

- $X'' \xrightarrow{\alpha} Y$ factors through Y' and $X'' \xrightarrow{\beta} Z$ factors through Z', and
- The following diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\alpha} & Y' \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ Z' & \xrightarrow{\delta} & W \end{array} \tag{2}$$

is a pullback diagram in **set**. (In particular the above (2) is a commutative diagram.)

Two coproducts

When do they give a Hopf monoid in species?

Consider decompositions

$$X = A \sqcup B \sqcup C \sqcup D.$$

Proposition (Part 1)

Let H *species in* **set**^{m} *with two comonoid structures* Δ^1 *and* Δ^2.

$$\begin{array}{ccc}
\mathrm{H}[X] & \xrightarrow{\Delta^2_{AC,BD}} & \mathrm{H}\begin{bmatrix} A \\ C \end{bmatrix} \Big| \, \mathrm{H}\begin{bmatrix} B \\ D \end{bmatrix} \\
\Big\downarrow \Delta^1_{AB,CD} & & \Delta^1_{A,C} \Big\downarrow \Delta^1_{B,D} \\
\dfrac{\mathrm{H}[A\,B]}{\mathrm{H}[C\,D]} & \xrightarrow[\Delta^2_{C,D}]{\Delta^2_{A,B}} & \begin{array}{c|c} \mathrm{H}[A] & \mathrm{H}[B] \\ \hline \mathrm{H}[C] & \mathrm{H}[D] \end{array} .
\end{array} \tag{3}$$

Proposition (Part II)

Let the product μ_2 be the dual map of Δ^2. Suppose maps are always partial maps.

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Let the product μ_2 be the dual map of Δ^2. Suppose maps are always partial maps. Then $\mathsf{H} = (\Delta^1, \mu_2, \epsilon, \eta)$ is a Hopf monoid in species iff the diagrams are always partial pull-back diagrams.

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How get Δ^1, Δ^2 ?

In most situations our species are species with restrictions:
Injections $Y \hookrightarrow X$ give restrictions

$$\begin{aligned} S[X] &\to S[Y] \\ s &\mapsto s_Y. \end{aligned}$$

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In most situations our species are species with restrictions:
Injections $Y \hookrightarrow X$ give restrictions

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Our coproducts shall then be of the form:

$$\begin{aligned} S[X] &\xrightarrow{\Delta} S[A] \times S[B] \\ s &\mapsto \begin{cases} \mathbf{0} \\ (s_A, s_B) \end{cases} \end{aligned}$$

Varying when $\Delta(s) = \mathbf{0}$ gives varying coproducts Δ.

Preorders

There is a species:

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Preorders

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$$\text{pre} : \textbf{set}^{\times} \to \textbf{set}$$
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- If P is a preorder P^{op} is the opposite preorder
- Preorders on X form a lattice. Given P, Q:

$$\text{Meet } P \wedge Q, \qquad \text{Join } P \vee Q.$$

Species from preorders

S is a species over preorders if it comes with a natural transformation

$$\mathsf{S} \xrightarrow{\pi} \mathsf{pre}$$

$$\mathsf{S}[X] \xrightarrow{\pi[X]} \mathsf{pre}[X]$$

$$s \mapsto \pi(s)$$

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Get coproduct

$$\mathsf{S}[X] \to \mathsf{S}[A] \times \mathsf{S}[B]$$

$$s \mapsto \begin{cases} (s_A, s_B), & (A, B) \text{ is a cut for} \pi(s) \\ \mathbf{0}, & (A, B) \text{ not cut for} \pi(s) \end{cases}$$

Pairs of restriction species over preorders:

$$\pi_1 : \mathsf{S} \to \mathsf{pre}, \quad \pi_2 : \mathsf{S} \to \mathsf{pre}$$

are intertwined restriction species over preorders if they fulfill the conditions in the pullback theorem.

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Part II

Examples of Hopf species/algebras

- Tensor and symmetric Hopf algebra
- Malvenuto-Reutenauer algebra
- Avoiding subspecies
- Hopf algebras of permutation avoidance: Loday-Ronco + +
- Hopf algebra of parking functions
- Hopf algebras of pairs of preorders

From Hopf species to Hopf algebras

The Fock functor

$$\overline{\mathcal{K}}(\mathsf{S}) = \oplus_{n \geq 0} \mathsf{k}\mathsf{S}[n]_{S_n}.$$

Tensor Hopf algebra

$k\langle x_1, \ldots, x_c \rangle$

- Product: Concatenation $x_5x_3 \cdot x_3x_6x_1 = x_5x_3^2x_6x_1$.
- Coproduct: Shuffle coproduct

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Comes from species:

- $C = \{1, \cdots, c\}$
- $S[X]$ = all pairs (s, T) where:
  - $s : X \to C$ is a coloring
  - T is a total order on X

Tensor Hopf algebra

- D the discrete preorder on X.
- S has two structures π_1, π_2 as species over preorders:

$$\pi_1(s, T) = D, \quad \pi_2(s, T) = T.$$

- Check: These give intertwined restriction species over preorders
- Δ^1 becomes coshuffling.

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- Δ^1 becomes coshuffling.
- Δ^2 becomes deconcatenation:

$$\mathrm{S}[X] \to \mathrm{S}[A] \times \mathrm{S}[B] \tag{4}$$

where $s \mapsto (s_A, s_B)$ if A is a down-set (initial segment) for the total order, and $s \mapsto 0$ if A is not an initial segment.

Symmetric Hopf algebra

$k[x_1, \ldots, x_c]$ with ordinary product and coproduct

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Δ^1 and Δ^2 are both the natural restriction maps

$$\mathsf{S}[X] \to \mathsf{S}[A] \times \mathsf{S}[B], \quad s \mapsto (s_A, s_B).$$

Malvenuto-Reutenauer Hopf algebra

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Restriction species with avoidance

Let S be a restriction species and A a subspecies (not necessarily a restriction species).

Definition

$S_{/A}[X]$ be those $s \in S[X]$ such that there is no injection $Y \hookrightarrow X$ such that $s_{|Y}$ in $A[X]$.

$S_{/A}$ is the A-avoiding subspecies of S.

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- S comonoid species $\Rightarrow S_{/A}$ comonoid species.

Malvenuto-Reutenauer Hopf algebra II

Recall:

- $S[X]$ = pairs (T_1, T_2) of total orders on X,

Malvenuto-Reutenauer Hopf algebra II

Recall:

- $S[X]$ = pairs (T_1, T_2) of total orders on X,
- S has two new structures ψ_1, ψ_2 as species over preorders:

$$\psi_1(s) = T_1 \vee T_2^{op}, \quad \psi_2(s) = T_1 \wedge T_2.$$

- Check: These give intertwined restriction species over preorders

This Hopf algebra is isomorphic to the Malvenuto-Reutenauer Hopf algebra.

Permutation avoiding subspecies

Let A be any set of permutations with no global descents. Let A be the subspecies consiting of permutations in A.

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Theorem

Then $\mathsf{S}_{/A}$ *is a Hopf species.*
It is a subcomonoid of S *w.r.t.* Δ^1 *and a quotient comonoid os* S *w.t.r.* Δ^2.

- When $A = \{213\}$ then $\mathrm{S}_{/A}$ gives the Loday-Ronco Hopf algebra or equivalently Foissy's Hopf non-commutative and non-cocommutative Hopf algebra of planar trees.
- When $A = \{213, 132\}$ one gets the Hopf algebra of quasi-symmetric functions.
- When $A = \{12\}$ one gets the polynomial ring in one variable.
- When $A = \{2413\}$ one gets $\cdots$

Parking functions

Definition

Let X a finite set with $|X| = n$. A function $p : X \to \mathbb{N}$ is a parking function if for each $i \leq n$, then $p^{-1}([1, i])$ has cardinality $\geq i$. Note this give: $p(X) \subseteq [1, n]$.

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- p induces a total preorder on X.
- p induces a total order on X iff $p : X \to [1, n]$ is a bijection.

Hopf algebra of parking functions

A master Hopf algebra

$S[X]$ = pairs of parking functions (p_1, p_2) on X.

- Induce two preorders
- Induce further two intertwining comonoid species.
- New Hopf species of parking functions, and a new Hopf algebra of parking functions.

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- Avoiding subspecies $S_{/A}$ consisting of pairs (p_1, p_2) where the p_2's correspond to total orders on X.
- Gives Hopf algebra of parking functions **PQSymm** due to Thibon et.al.

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They fall into three classes. Examples are:

- Class I: Malvenuto-Reutenauer
- Class II: Quasi-symmetric functions
- Class III: Symmetric funcions

Thank you!

What did we learn?

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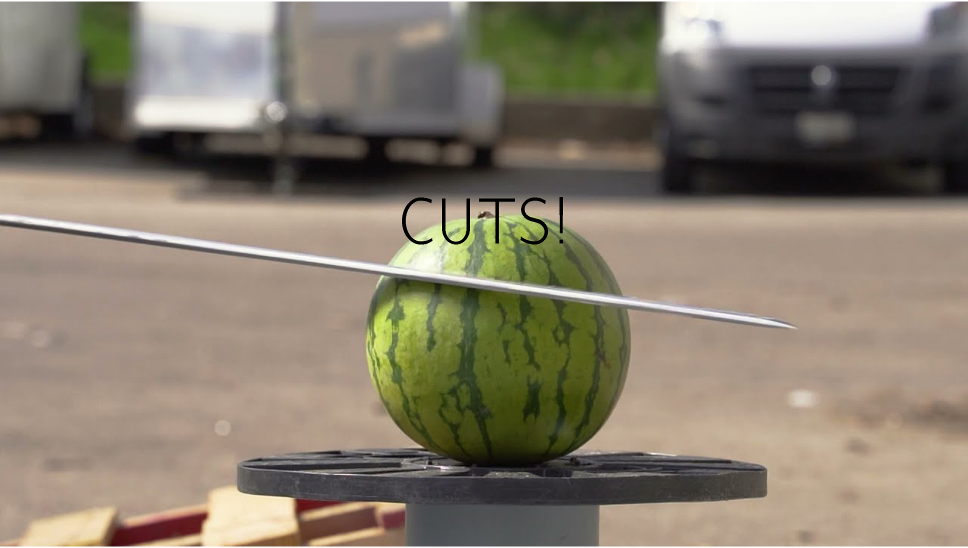
Combinatorial Hopf algebras come from $\cdots$


CUTS!