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Bilateral Exchange and Competitive Equilibrium

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In honour of Professor Lionel Thibault

Abstract Motivated by computerized markets, this paper considers direct exchange between matched agents, just two at a time. Each party holds a "commodity vector," and each seeks, whenever possible, a better holding. Focus is on feasible, voluntary exchanges, driven only by (projected) differences in generalized gradients.

The paper plays down the importance of agents' competence, experience and foresight. It also reduces the role of optimization, and it allows non-smooth data. Yet it identifies reasonable conditions which suffice for convergence to competitive equilibrium.

Keywords: bilateral exchange - convex preferences - competitive equilibrium - generalized gradients - transferable utility.

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1 Introduction

Bilateral exchange is the oldest mode of trade [3]. Though *direct and non-anonymous*, it entails numerous problems - be it with bargaining, matching, pricing, quantity, quality, or search [17], [21], [22]. In contrast, when mediated by markets and money, *indirect and anonymous* transactions bring numerous advantages [23]. It's interesting therefore that new institutions and platforms have emerged - many during recent decades - which combine properties of both arrangements. To wit, most internet-based exchanges are now direct *yet* anonymous [19]. Hybrid systems, which feature dealers and specialists, may operate likewise. Their rationale remains the traditional one: to translate latent demand-supply into realized prices and quantities.

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This paper models bilateral trades, undertaken time and again, between diverse economic agents. Read as a tale of exchange, it's easily dismissed as reductionist mathematics because it ignores the cultural richness of transactions.[1]

This notwithstanding, the model below emphasizes that most agents lack some foresight or knowledge. Whence they adapt repeatedly - with caution or moderation - to changing circumstances.[2] In differential terms, agents' perceptions of their planning problems are limited or localized - and merely of first order. Those features are handicaps - and more so if choice be constrained and objectives non-smooth. Nonetheless, suppose agents behave *as though* they subscribe to selected parts of convex and set-valued analysis. Specifically, suppose generalized gradients - and in particular, their feasible components - guide exchanges. On such premises, *might non-coordinated agents, by themselves, make equilibrium prices emerge*?

To address that question, Section 2 first considers just *one* direct exchange, somewhat modest and myopic, between merely two agents, recently matched on a common platform. Section 3 shows, under rather weak assumptions, that repeated exchanges may entail convergence to competitive equilibrium. Section 4 outlines two examples, and Section 5 concludes.

2 Bilateral Exchange

Consider an economic agent i who actually owns a "commodity vector" $x_i \in X_i$. The set X_i, which accounts for his constraints, is presumed closed convex. It's part of an ambient real Euclidean space $\mathbb{X}$, endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$.[3] If agent i contends with x_i, he obtains payoff or transferable utility $u_i(x_i)$, his criterion $u_i : \mathbb{X} \to \mathbb{R}$ being concave.

For interpretation, construe i as a producer who would get revenue $u_i(x_i)$ from factor bundle x_i. The components of x_i could specify rights to use various resources - say, water, fish quotas, pollution permits, or land; see Example 1. Alternatively, x_i could be an insurance policy or a financial security; see Example 2. In either example, x_i is a *contract*, written on various goods, states or times. While the latter items might be perishable or transient, the contract stands. In short, exchange need not proceed in kind, but rather in user rights and payments. But plainly, the subsequent analysis does not hinge upon any such specific interpretation of x_i.

Suppose agent i meets - or is matched with - another economic agent j. The latter holds some vector x_j in a closed convex set $X_j \subseteq \mathbb{X}$. He

[1] In view of developments in experimental economics [24], this paper might be construed as a mathematically inclined narrative of multi-agent, market-based betterment.

[2] Some theorists declare such agents *boundedly rational*. Quite often, however, the real issue is rather agents' lack of experience or information.

[3] $\mathbb{X}$ is finite-dimensional here, but results extend to Hilbert space settings.

worships maximization of his concave payoff $u_j : \mathbb{X} \to \mathbb{R}$ over X_j. Most likely, however, neither he nor his interlocutor is perfectly well versed in optimization. And either might lack some foresight and information. For such or other reasons, both proceed with caution and moderation. What commodity transfer $\Delta \in \mathbb{X}$, to i from j, might then be modest, yet desirable and feasible for both parties?

Beginning with feasibility, suppose no quantity of any commodity be created or destroyed during any feasible transfer. If the updated holding x_i^{+1} of agent i differs from x_i, let $d := (x_i^{+1} - x_i)/\left\|x_i^{+1} - x_i\right\|$ denote the corresponding direction of transferal - and $\sigma := \left\|x_i^{+1} - x_i\right\|$ the associated step-size. Consequently, with no loss of generality, after transfer $\Delta = x_i^{+1} - x_i = \sigma d$, to i from j, the updated positions become

$$x_i^{+1} = x_i + \sigma d \in X_i \quad \text{and} \quad x_j^{+1} = x_j - \sigma d \in X_j, \tag{1}$$

with $\sigma \geq 0$ and $\|d\| \leq 1$. Moreover, $\sigma d \neq 0 \Longrightarrow \|d\| = 1$. The first inclusion in (1) implies that d belongs to the convex cone

$$D_i(x_i) := \{r(\chi_i - x_i) \mid r \in \mathbb{R}_+ \ \& \ \chi_i \in X_i\} =: \mathbb{R}_+(X_i - x_i),$$

composed of all feasible directions for agent i at x_i. Similarly, the second inclusion in (1) tells that $-d \in D_j(x_j) = \mathbb{R}_+(X_j - x_j)$. In short, the interlocutors must choose a direction

$$d \in D_{ij}(x_i, x_j) := D_i(x_i) \cap -D_j(x_j).$$

This requirement on d explains the first part in the following proposition. For the second part there, declare $g_i \in \mathbb{X}$ a *supergradient* of u_i at x_i, and write $g_i \in \partial u_i(x_i)$, iff

$$u_i(\chi_i) \leq u_i(x_i) + \langle g_i, \chi_i - x_i \rangle \quad \text{for all} \quad \chi_i \in \mathbb{X}.$$

Further, at any $x_i \in X_i$, let

$$N_i(x_i) := \{n_i \in \mathbb{X} \mid \langle n_i, X_i - x_i \rangle \leq 0\}$$

denote the outward *normal cone* there.

Proposition 2.1 (On bilateral trade). *When agents i, j own respectively $x_i \in X_i$ and $x_j \in X_j$, they* ***cannot*** *make a proper trade in case the cone of feasible directions $D_{ij}(x_i, x_j)$ is degenerate. Moreover, they* ***ought not*** *make any trade if*

$$[\partial u_i(x_i) - N_i(x_i)] \cap [\partial u_j(x_j) - N_j(x_j)] \neq \varnothing.$$

The reason is that (1) *then yields* $u_i(x_i^{+1}) + u_j(x_j^{+1}) \leq u_i(x_i) + u_j(x_j)$.

Proof. For the second statement, suppose $g_i \in \partial u_i(x_i)$, $g_j \in \partial u_j(x_j)$ and

$n_i \in N_i(x_i)$, $n_j \in N_j(x_j)$ satisfy $g_i - n_i = g_j - n_j$. Together, $x_i^{+1} = x_i + \sigma d$ and $d \in D_i(x_i)$ imply

$$u_i(x_i^{+1}) \leq u_i(x_i) + \sigma \langle g_i, d \rangle \leq u_i(x_i) + \sigma \langle g_i, d \rangle - \sigma \langle n_i, d \rangle .$$

Quite likewise, $x_j^{+1} = x_j + \sigma(-d)$ and $-d \in D_j(x_j)$ imply

$$u_j(x_j^{+1}) \leq u_j(x_j) - \sigma \langle g_j, d \rangle \leq u_j(x_j) - \sigma \langle g_j, d \rangle + \sigma \langle n_j, d \rangle .$$

Adding theses inequalities yields

$$u_i(x_i^{+1}) + u_j(x_j^{+1}) \leq u_i(x_i) + u_j(x_j) + \sigma \langle g_i - n_i, d \rangle - \sigma \langle g_j - n_j, d \rangle = u_i(x_i) + u_j(x_j).$$

This completes the proof. □

For brevity and interpretation, call any

$$p_i \in g_i - n_i \quad \text{with} \quad g_i \in \partial u_i(x_i) \quad \text{and} \quad n_i \in N_i(x_i)$$

an essential margin or *price* used by agent i at x_i. He applies it only there to have an idiosyncratic local valuation of marginal transfers.[4]

For a backdrop, recall the Walrasian narrative about tâtonnement in prices. It features some fictive resource custodian who adjusts *common prices* so as finally to ensure material balances.[5] Here, in contrast, materials always balance, and differences in *personal prices* drive all deals.

Precisely to the point, Proposition 2.1 tells that agents i and j ought not trade when they see equal prices $p_i = p_j$. This observation already indicates possible avenues towards equilibrium, namely: after repeated bilateral barters, some common price (vector) should emerge. Prior to such emergence, before all personal prices coincide, if two agents - say, i and j - still see no common price, what feasible transfer might suit both? Proposition 2.1 directs attention to instances where

$$D_{ij}(x_i, x_j) \neq \{0\} \quad \text{and} \quad [\partial u_i(x_i) - N_i(x_i)] \cap [\partial u_j(x_j) - N_j(x_j)] = \varnothing. \tag{2}$$

To clarify that such a setting invites trade, use directional derivatives

$$u_i'(x_i; d) := \lim_{\sigma \to 0^+} \frac{u_i(x_i + \sigma d) - u_i(x_i)}{\sigma}$$

to define the *steepest slope* of the agents' joint payoff $u_i + u_j$:

$$\mathfrak{S}_{ij}(x_i, x_j) := \sup \left\{ u_i'(x_i; d) + u_j'(x_j; -d) \;\mid\; d \in D_{ij}(x_i, x_j) \;\&\; \|d\| \leq 1 \right\}. \tag{3}$$

[4] Since a normal component $n_i \in N_i(x_i)$ of any "gradient" $g_i \in \partial u_i(x_i)$ points right out of X_i, it's subtracted to leave a personal price $p_i = g_i - n_i$. If the resulting p_i belongs to $clD_i(x_i)$, then $\langle p_i, N_i(x_i) \rangle \leq 0$, and $\langle p_i, d \rangle \geq \langle g_i, d \rangle$ for each $d \in D_i(x_i)$.

[5] The Danzig-Wolfe planning procedure is of similar flavor [5].

Can the said slope characterize situation (2)? Indeed, it can. Another description of $\mathfrak{S}_{ij}(x_i, x_j)$ helps to explain how. It invokes the closed *tangent cone* $T_{ij}(x_i, x_j) := clD_{ij}(x_i, x_j)$ and the *orthogonal projection*

$$P_{ij}[v] := \arg\min\{\|v - t\| \mid t \in T_{ij}(x_i, x_j)\}$$

onto that cone.[6]

Proposition 2.2 (The steepest slope as the norm of a projected gradient difference). *The steepest slope* (3) *equals*

$$\begin{aligned}\mathfrak{S}_{ij}(x_i, x_j) &= \max\left\{u_i'(x_i; d) + u_j'(x_j; -d) \mid d \in T_{ij}(x_i, x_j) \ \& \ \|d\| \le 1\right\} \\ &= \min\{\|P_{ij}[g_i - g_j]\| \mid g_i \in \partial u_i(x_i),\ g_j \in \partial u_j(x_j)\}.\end{aligned} \tag{4}$$

Proof. Recall that a concave function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ which is finite near $x \in \mathbb{X}$, has a non-empty compact convex *superdifferential* $\partial f(x)$ and a directional derivative

$$f'(x; d) = \lim_{r \to 0^+} \frac{f(x + rd) - f(x)}{r} = \min\{\langle x^*, d\rangle \mid x^* \in \partial f(x)\}; \tag{5}$$

see Theorem 2.87 in [15]. Since $f'(x; d)$ is concave in d, it's continuous in that variable. This continuity justifies two replacements in definition (3): first, $T_{ij}(x_i, x_j)$ for $D_{ij}(x_i, x_j)$, and second, maximum for supremum. This takes care of the leading equality in (4). The last one there follows from $\mathfrak{S}_{ij}(x_i, x_j) =$

$$\max_d \min_{g_i, g_j}\{\langle g_i - g_j, d\rangle \mid g_i \in \partial u_i(x_i),\ g_j \in \partial u_j(x_j),\ d \in T_{ij}(x_i, x_j) \ \& \ \|d\| \le 1\}$$

$$= \min_{g_i, g_j} \max_d \{\langle g_i - g_j, d\rangle \mid g_i \in \partial u_i(x_i),\ g_j \in \partial u_j(x_j),\ d \in T_{ij}(x_i, x_j) \ \& \ \|d\| \le 1\}$$

$$= \min\{\|P_{ij}[g_i - g_j]\| \mid g_i \in \partial u_i(x_i),\ g_j \in \partial u_j(x_j)\}.$$

In the preceding string, the first equality uses definition (3) and formula (5). Since all sets $\partial u_i(x_i)$, $\partial u_j(x_j)$, and $T_{ij}(x_i, x_j) \cap$ (unit ball $\mathbb{B}$) are non-empty compact convex, the second equality follows from von Neumann's minmax theorem.

For the last equality, recall the following result of Moreau: Given a non-empty closed convex cone $T \subseteq \mathbb{X}$, any vector $v \in \mathbb{X}$ has a unique *orthogonal decomposition* $v = t + n$ into a *"tangent"* $t \in T$, and a *"normal"* $n \in N := \{n \in \mathbb{X} \mid \langle n, T\rangle \le 0\}$, the two components being perpendicular: $\langle t, n\rangle = 0$; see Thm. 3.8 in [14]. Since the cone $clD_{ij}(x_i, x_j)$ is indeed closed convex, let $v = g_i - g_j$ and $t = P_{ij}[v]$. Now the last equality in the above

[6] I ought write $P_{ij}[\cdot, x_i, x_j]$ for the operator $P_{ij}[\cdot]$. The pair (x_i, x_j) is, however, tacitly understood or clear from the context.

string derives from the Cauchy-Schwartz inequality, using the Moreau decomposition of v with respect to $T_{ij}(x_i, x_j)$. □

In the convenient case, when $(x_i, x_j) \in int(X_i \times X_j)$,

$$\mathfrak{S}_{ij}(x_i, x_j) = \min\{\|g_i - g_j\| \mid g_i \in \partial u_i(x_i),\ g_j \in \partial u_j(x_j)\}.$$

If moreover, u_i and u_j are differentiable at x_i and x_j respectively, $\mathfrak{S}_{ij}(x_i, x_j) = \left\|u_i'(x_i) - u_j'(x_j)\right\|$. Thus, it's expedient that some agent has a smooth objective or invariably makes an interior choice; see Theorem 3.2. Anyway, when $\mathfrak{S}_{ij}(x_i, x_j) > 0$, prospects appear promising for improvement of the joint payoff $u_i(x_i) + u_j(x_j)$. In fact, as seen next, the steepest slope is strictly positive iff situation (2) prevails.

Proposition 2.3 (Positive slope and joint improvement). *The steepest slope has the alternative expression*

$$\mathfrak{S}_{ij}(x_i, x_j) = \inf\{\|p_i - p_j\| \mid p_i \in \partial u_i(x_i) - N_i(x_i)\ \&\ p_j \in \partial u_j(x_j) - N_j(x_j)\}. \tag{6}$$

It follows that $\mathfrak{S}_{ij}$ is lower semicontinuous on $X_i \times X_j$. Further, $\mathfrak{S}_{ij}(x_i, x_j) > 0$ iff (2) prevails. In fact, $\mathfrak{S}_{ij}(x_i, x_j) = 0$ iff x_i, x_j already

$$maximize\ u_i(x_i^{+1}) + u_j(x_j^{+1})\ \ s.\ t.\ (x_i^{+1}, x_j^{+1}) \in X_i \times X_j\ \&\ x_i^{+1} + x_j^{+1} = x_i + x_j, \tag{7}$$

or equivalently, iff

$$0 \in P_{ij}\left[\partial u_i(x_i) - \partial u_j(x_j)\right].$$

Proof. Bauschke and Borwein [1] have already considered the case where $\partial u_i(x_i)$ and $\partial u_j(x_j)$ are singletons. For the more general result, define the *distance* between two sets $\mathcal{C}_i$, $\mathcal{C}_j \subset \mathbb{X}$ by

$$dist[\mathcal{C}_i, \mathcal{C}_j] := \inf\|\mathcal{C}_i - \mathcal{C}_j\| = \inf\{\|c_i - c_j\| \mid c_i \in \mathcal{C}_i,\ c_j \in \mathcal{C}_j\}.$$

In these terms, the right hand side of (6) equals

$$dist[\partial u_i(x_i) - N_i(x_i), \partial u_j(x_j) - N_j(x_j)]$$

$$= \inf\{dist[g_i - N_i(x_i), g_j - N_j(x_j)] \mid g_i \in \partial u_i(x_i), g_j \in \partial u_j(x_j)\}$$

$$= \min\{\|P_{ij}(g_i - g_j)\| \mid g_i \in \partial u_i(x_i), g_j \in \partial u_j(x_j\}.$$

For the attainment of the minimal distance note that $\partial u_i(x_i)$ and $\partial u_j(x_j$ are compact convex. The lower semicontinuity of $\mathfrak{S}_{ij}$ derives from the fact that the correspondences $x_i \rightrightarrows \partial u_i(x_i)$, $N_i(x_i)$ are (upper) outer semicontinuous [20]. □

In order to account for individual rationality, Feldman (1973) added the constraints $u_i(x_i^{+1}) \geq u_i(x_i)$ and $u_j(x_j^{+1}) \geq u_j(x_j)$ to problem (7). Thus he required that any outcome of bilateral barter be a *core solution of a* *two-person cooperative game.* However, to find such an outcome, the parties need considerable information and skill - typically more than depicted in their above portraits.

Feldman also presumed that each set X_i be the nonnegative orthant, and that each u_i come continuously differentiable. Like [10], this paper relaxes these restrictions.[7] Unlike [10], no penalty functions are invoked, and feasibility is maintained throughout. In addition, this paper matches players by a different protocol, and it dispenses with some intrinsic assumptions of [10], hard to verify.

I emphasize that some objective u_i or u_j could well be nonsmooth (see Example 1) - and that orthogonal projection isn't always easy. Consequently, it can require some competence or effort of agents i, j to identify the steepest slope and an associated best direction. Assigning prominence to these two objects doesn't quite square with this paper's chief purpose, namely: to provide a low-complexity model of direct deals. Queries of this sort motivate a relaxed transaction, one for which (3) is realized at least up to a fixed fraction $\varphi_{ij} \in (0, 1)$.[8]

Definition (Real transfer). *Agents i, j make a* ***real transfer*** *if* (1) *holds with $d \in D_{ij}(x_i, x_j)$, $\|d\| = 1$, and*

$$\Delta u_{ij} := u_i(x_i^{+1}) + u_j(x_j^{+1}) - u_i(x_i) - u_j(x_j) \geq \sigma\varphi_{ij}\mathfrak{S}_{ij}(x_i, x_j) > 0. \quad (8)$$

Proposition 2.4 (On real transfers). *Whenever $(x_i, x_j) \in X_i \times X_j$ and $\mathfrak{S}_{ij}(x_i, x_j) > 0$, agents i, j may indeed make a real transfer. Then, the inequalities*

$$u_i(x_i^{+1}) + m_i > u_i(x_i) \quad \text{and} \quad u_j(x_j^{+1}) + m_j > u_j(x_j)$$

are solvable with monetary side-payments m_i, m_j that sum to zero.

Proof. Let $\mathfrak{S}_{ij}(x_i, x_j; d) := u_i'(x_i; d) + u_j'(x_j; -d)$. By definition (3) there exists $d \in D_{ij}(x_i, x_j)$, $\|d\| \leq 1$, *such that* $\mathfrak{S}_{ij}(x_i, x_j; d) \geq \varphi_{ij}^{1/2}\mathfrak{S}_{ij}(x_i, x_j)$. The positive homogeneity of the directional derivatives ensures that the agents may take $\|d\| = 1$. For small enough $\sigma > 0$, inclusions (1) hold and $\Delta u_{ij} \geq \sigma\varphi_{ij}^{1/2}\mathfrak{S}_{ij}(x_i, x_j; d)$. Combining the last two inequalities, (8) follows forthwith.

If only $u_i(x_i^{+1}) > u_i(x_i)$, offer agent j money m_j subject to $u_j(x_j) - u_j(x_j^{+1}) < m_j < u_i(x_i^{+1}) - u_i(x_i)$. Thereafter debit agent i that same amount. □

In short, modulo suitable side payments, real transfers generate strict Pareto

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[7] Except that if a feasible allocation (x_i) is pairwise efficient, the essential margin $\partial u_i(x_i) - N_i(x_i)$ should reduce to a singleton for at least one agent i; see Theorem 3.2.

[8] The attending stepsize selection relates to the *Goldstein rule*.

improvement. Money "oils" the transaction machinery. Deals and incentives are compatible. Nothing is said here, however, about how agents i, j divide $\Delta u_{ij} > 0$. All issues about bargaining are deliberately left open [17], [22].

3 Convergence to Competitive Equilibrium

Decision theory and economics often make overly strong demands on agents' competence and rationality. It's therefore tempting to ask: Could adaptive behavior partly serve as Ersatz for competence? May iterated market transactions substitute for perfect rationality [13]? This section indicates positive answers to both questions. It is concerned with emergence of a market clearing price system.

Accommodated henceforth is a finite set I of economic agents. At the outset, individual $i \in I$ already holds some *endowment* $e_i \in X_i$. A profile $\mathbf{x} = (x_i) \in \mathbb{X}^I$ is called an *allocation* iff $\sum_{i \in I} x_i = \sum_{i \in I} e_i$. It is *feasible* moreover, iff $x_i \in X_i$ for each i.

To organize arguments, it's expedient to view the exchange process unfold like an *algorithm,* fictitious or real, but affected by some *protocol* that decides who will trade next with whom. By tacit assumption, no auctioneer or referee comes to the fore - and no central planner, coordinator, or invisible hand works backstage.

Repeated bilateral barters construed as an algorithm:
- *Start* with some feasible allocation.
- *Invoke the protocol* to activate or match two agents.
- *If their steepest slope* (3) *is nil, invoke the protocol* anew. *Stop* only when all steepest slopes vanish.
- *Otherwise, the active agents make a real transfer* (8).
- *Continue* to *invoke the protocol* until convergence.

Thus, at discrete stages $k = 0, 1, ...$ two selected agents make a real transfer.
Stopping is idealized and too stringent here. In practice, exchange terminates, and the market settles, when all $\mathfrak{S}_{ij}(x_i, x_j)$ are negligible or so small as to pass unnoticed.
Protocols can be manifold. There is room for random pairing, deliberate search, asynchronous or parallel matching - and for different affinities among agents. Broadly, what imports is that each agent pair be activated repeatedly [9], [10], [12]. Here I shall only presume that *periodically, the pair that trades enjoys maximal steepest slope.*

Proposition 3.1 (Exhaustion of two-sided trade options). *Suppose that periodically, agents with maximal steepest slope* (3) *make real transfers* (8). *Also suppose trading agents* i, j *use a step-size* $\sigma_{ij}(x_i, x_j) \geq \underline{\sigma}_{ij}(x_i, x_j)$ *where* $\underline{\sigma}_{ij}$ *is lower semicontinuous and positive wherever* $\mathfrak{S}_{ij}(x_i, x_j) > 0$.

Then, all steepest slopes are nil at every accumulation point of the resulting sequence.

Proof. Let $\mathbf{A}$ denote the set of all feasible allocations. Its subset

$$\mathbf{A}_0 := \{\mathbf{x} = (x_i) \in \mathbf{A} \mid \text{all } \mathfrak{S}_{ij}(x_i, x_j) = 0\} \tag{9}$$

is of prime interest here. For $\mathbf{x} \in \mathbf{A}$, posit $U(\mathbf{x}) := \sum_{i \in I} u_i(x_i)$. Let $\mathbf{d}_{ij} \in \mathbb{X}^I$ have all components 0 except i, j which feature unit vectors d_i and d_j respectively such that $d_i + d_j = 0$. With reference to real transfers (8), define $\varphi := \min_{ij} \varphi_{ij}$. At each profile $\mathbf{x} \in \mathbf{A} \diagdown \mathbf{A}_0$ let $\mathfrak{S}(\mathbf{x}) := \max_{i,j} \mathfrak{S}_{ij}(x_i, x_j)$ and

$$\mathbf{B}(\mathbf{x}) := \left\{\mathbf{x}^{+1} = \mathbf{x} + \sigma_{ij}\mathbf{d}_{ij} \in \mathbf{A} \;\middle|\; \sigma_{ij} \geq \underline{\sigma}_{ij}(x_i, x_j) \text{ and } U(\mathbf{x}^{+1}) \geq U(\mathbf{x}) + \sigma_{ij}\varphi\mathfrak{S}(\mathbf{x}) \right\}.$$

In contrast, when $\mathbf{x} \in \mathbf{A}_0$, let $\mathbf{B}(\mathbf{x}) = \{\mathbf{x}\}$. The point-to-set correspondence $\mathbf{B} : \mathbf{A} \rightrightarrows \mathbf{A}$, so defined, is closed outside $\mathbf{A}_0$. Moreover, if $\mathbf{x} \notin \mathbf{A}_0$ and $\mathbf{x}^{+1} \in \mathbf{B}(\mathbf{x})$, then $U(\mathbf{x}^{+1}) > U(\mathbf{x})$. The conclusion now follows from Theorem 7.3.4. in [2] - an extension of the Zangwill convergence theorem. □

The conditions imposed on $\underline{\sigma}_{ij}$ secure that trading agents i, j strictly improve their payoffs when indeed they can. A chief issue remains, however. Will agents reach the "equilibrium" subset $\mathbf{E} \subseteq \mathbf{A}_0$, composed of all feasible allocations $\mathbf{x} = (x_i)$ at which a common price prevails? Such a price must belong to

$$\cap_{i \in I} [\partial u_i(x_i) - N_i(x_i)] \neq \emptyset. \tag{10}$$

Alas, inclusion $\mathbf{E} \subseteq \mathbf{A}_0$ can be strict:

Example 0 (*Complete trade with no common price*). Consider three agents $i \in \{1, 2, 3\}$ who exchange two goods, labelled $s \in S = \{1, 2\}$. Posit $\mathbb{X} = \mathbb{R}^S$, write $x_{is} := x_i(s)$, and use utility functions

$$\boxed{\begin{aligned} u_1(x_1) &= \min\{x_{11}, 0\}, \\ u_2(x_2) &= \min\{0, x_{22}\}, \\ u_3(x_3) &= \min\{x_{31}, x_{32}\}. \end{aligned}}$$

Let each $X_i = [-1, 1]^2$. At $x_i = e_i := (0, 0)$ each cone $N_i(x_i)$ is degenerate, and the agents have superdifferentials

$$\partial u_1(x_1) = [(1, 0), (0, 0)], \quad \partial u_2(x_2) = [(0, 0), (0, 1)], \quad \partial u_3(x_3) = [(1, 0), (0, 1)].$$

Consequently, $\partial u_i(x_i) \cap \partial u_j(x_j) \neq \emptyset$ for each agent pair i, j - that is, each $\mathfrak{S}_{ij}(x_i, x_j) = 0$ so that $\mathbf{x} \in \mathbf{A}_0$ - but $\mathbf{x} \notin \mathbf{E}$ because $\cap_{i \in I} \partial u_i(x_i)$ is empty.[9] ♢

In an important exception, fitting trade of CO_2-emission rights, $\mathbb{X}$ is one-dimensional. Equality $\mathbf{A}_0 = \mathbf{E}$ then follows from Helly's theorem in that

[9] Allocation $(0, -1)$, $(-1, 0)$, $(1, 1)$ is efficient, and $p = (0, 0)$ is an equilibrium price.

$[\partial u_i(x_i) - N_i(x_i)] \cap [\partial u_j(x_j) - N_j(x_j)] \neq \varnothing \ \forall i, j \Longrightarrow$ (10). Hence $\mathbf{A}_0 = \mathbf{E}$.

The following main result synthesizes the preceding facts:

Theorem 3.2 (Convergence to competitive equilibrium). *Let the set of feasible allocations be bounded. Suppose that at every* $(x_i) \in \mathbf{A}_0$ (9), *at least one agent* i *has* x_i *interior to* X_i *and* u_i *Gâteaux differentiable at that point. Then, under the hypotheses of Proposition 3.1, the exchange process accumulates, and each limit point* (x_i) *is an equilibrium profile in that demand equals supply:* $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, *and there is a common price* p *under which each agent has maximized his total payoff*:

$$u_i(\chi_i) + \langle p, e_i - \chi_i \rangle \leq u_i(x_i) + \langle p, e_i - x_i \rangle \quad \forall \chi_i \in X_i \ \forall i \in I. \tag{11}$$

Proof. Since the feasible set is compact, repeated trade leads to at least one accumulation point $\mathbf{x} = (x_i)$. Any such point is a feasible allocation. Moreover, Proposition 3.1 implies that $\mathbf{x} \in \mathbf{A}_0$ (9). By assumption, some agent i has $x_i \in intX_i$ and $\partial u_i(x_i)$ reduced to a singleton. That is, the (local) price for this agent equals his (unique) gradient $u_i'(x_i) =: p \in \partial u_i(x_i) - N_i(x_i)$. Since $\mathfrak{S}_{ij}(x_i, x_j) = 0$ for all $j \neq i$, Proposition 2.3 yields that

$$\{p\} = [\partial u_i(x_i) - N_i(x_i)] \cap [\partial u_j(x_j) - N_i(x_j)] \quad \forall\ j \neq i,$$

hence $p \in \cap_{i \in I} [\partial u_i(x_i) - N_i(x_i)]$. Finally, considering *any* agent i, the inclusions $g_i \in \partial u_i(x_i)$, $n_i \in N_i(x_i)$, $\chi_i \in X_i$, and the relation $p = g_i - n_i$ imply

$$u_i(\chi_i) \leq u_i(x_i) + \langle g_i, \chi_i - x_i \rangle \leq u_i(x_i) + \langle g_i - n_i, \chi_i - x_i \rangle = u_i(x_i) + \langle p, \chi_i - x_i \rangle,$$

from which (11) follows forthwith. □

4 Two Examples

This section brings out two fairly general instances. One concerns linear production games [18]; the other deals with trade of contingent claims under asymmetric information [7]. In either instance the cone $D_i(x_i)$ of feasible directions is closed convex, and a best direction is easily found.

Example 1: Linear production economies [12]. Let

$$u_i(x_i) := \sup \{ y_i^* \cdot y \mid x_i \geq A_i y_i \ \ \& \ \ y_i \geq 0 \}. \tag{12}$$

Here the "price-vector" y_i^* and the "activity plan" y_i both belong to a Euclidean space $\mathbb{Y}_i$. That space and $\mathbb{X}$ are equipped with standard vector orders and inner products. The linear mapping $A_i : \mathbb{Y}_i \to \mathbb{X}$ represents a technology that consumes various production factors - of which the bundle x_i is available.[10]

[10] Agent i's feasible set $X_i := u_i^{-1}(\mathbb{R})$.

By linear programming duality, $x_i^* \in \partial u_i(x_i)$ iff x_i^* solves the dual to problem (12), namely:

$$\inf \left\{ x_i^* \cdot x_i \;\middle|\; A_i^T x_i^* \geq y_i^* \;\;\&\;\; x_i^* \geq 0 \right\}.$$

Moreover, the cone $D_i(x_i)$ is closed convex and easily computable at any feasible x_i. Indeed, if $\mathbb{X} = \mathbb{R}^S$ for some finite list S of goods, the active index ensemble

$$S_i(x_i) := \{ s \in S \mid [x_i - A_i y_i]_s = 0 \text{ and } y_i \text{ is optimal in (12)} \}$$

identifies the binding constraints. Then $D_i(x_i)$ equals the "orthant" that has $\mathbb{R}_+$ in each component $s \in S_i(x_i)$, and the entire line $\mathbb{R}$ in all others. Hence projection onto $D_{ij}(x_i, x_j)$ is easily executed.

In principle, extensions to non-linear production games are immediate. Specifically, in analogy with (12), let the reduced utility

$$u_i(x_i) := \sup \{ f_i(y_i) \mid x_i \geq g_i(y_i) \;\;\&\;\; y_i \in Y_i \}$$

stem from convex functions $-f_i : \mathbb{Y}_i \to \mathbb{R}$, $g_i : \mathbb{Y}_i \to \mathbb{X}$, and a closed convex set $Y_i \subseteq \mathbb{Y}$. Then, to find supergradients $x_i^* \in \partial u_i(x_i)$ - alias *Lagrange multipliers* - is usually harder. But again, to identify $D_i(x_i)$ amounts only to assess which constraints are binding.

Example 2: Risk Exchange [7], [8], [11]. In the setting of finance or insurance, plagued by uncertainty about the future, let S denote a finite full set of relevant, but mutually exclusive states $s \in S$. Posit $\mathbb{X} = \mathbb{R}^S$ as the linear space of all *contingent claims* $x : S \to \mathbb{R}$ to money.

Such a claim $x \in \mathbb{X}$ is *adapted* to a *partition* $\mathcal{P}$ of S, written $x \in \mathcal{A}(\mathcal{P})$, if x is constant on each part $P \in \mathcal{P}$. Agent i is *information constrained* iff $X_i \subseteq \mathcal{A}(\mathcal{P}_i)$ for some proper partition $\mathcal{P}_i$ of S. In particular, when $X_i = \mathcal{A}(\mathcal{P}_i)$, agent i can identify state s ex post only up to the part $P(s) \in \mathcal{P}_i$ which contains s. Agents i, j have *asymmetric information* structures if $\mathcal{P}_i \neq \mathcal{P}_j$; see [7].

Of particular importance are instances where $X_i = \mathcal{A}(\mathcal{P}_i)$ for each $i \in I$. Then, since $\mathcal{A}(\mathcal{P}_i)$ is a closed linear subspace, each direction $d \in D_{ij}(x_i, x_j) = X_i \cap X_j$ must be constant on $P_i \cup P_j$ whenever the parts $P_i \in \mathcal{P}_i$ and $P_j \in \mathcal{P}_j$ intersect.

For this reason, many agent pairs i, j may see few feasible barters of mutual interest. In contrast, suppose a particular party j enjoys the most fine-grained information, his state-space partition $\mathcal{P}_j$ being composed only of singletons. Since $X_j = \mathbb{X}$, presence of such a well-informed "intermediary" or broker j largely facilitates trade.

As to computation, suppose scenario $s \in S$ comes up with objective probability $\pi_s > 0$, $\sum_{s \in S} \pi_s = 1$. Endow $\mathbb{X}$ with probabilistic inner product $\langle x^*, x \rangle := \sum_{s \in S} x_s^* x_s \pi_s$ and corresponding norm. Then, on part P of a partition $\mathcal{P}$, the projection $\Pr(x)$ of $x \in \mathbb{X}$ onto the subset $X := \mathcal{A}(\mathcal{P})$ equals the *conditional expectation*

$$\Pr(x)_s = \frac{\sum_{s \in P} x_s \pi_s}{\sum_{s \in P} \pi_s} \quad \text{for each} \quad s \in P. \ \Diamond$$

5 Concluding Remarks

I have, with no excuse, accommodated agents of "bounded rationality," each with a local, limited perspective. Chief concerns were with asymptotic stability of the market. Complexity and convergence rates remain secondary issues.[11] No claims were made in that regard, and stages were not related to real time.

As modelled above, repeated exchanges invite different interpretations. One construes the process as pure fiction about (Marshallian) tâtonnement, featuring trials and errors in quantities. In that optic, what comes on stage is a hypothetical preludium, played out prior to any proper exchange. In a second interpretation, the story just serves to motivate various slow, fully decentralized algorithms. Both views are coherent and respectable, but I share neither. Real agents do trade out of equilibrium.[12]

It deserves emphasis that most markets display a permanent *bid-ask spread* no less than some monetary tick $\varepsilon > 0$. So, for practical purposes, what replaces the commonality of prices (10) is the relaxed version

$$\cap_{i \in I} \left[\partial u_i(x_i) - N_i(x_i) + \varepsilon \mathbb{B} \right] \neq \varnothing,$$

$\mathbb{B}$ being the closed unit ball. Any p from the above intersection qualifies as approximate equilibrium price. Upon allowing such "fuzzy" prices, convergence may prove fairly rapid - possibly finite.

I have also ignored that some constraints could be implicit. To wit, only finite-valued functions u_i were admitted here above. Some instances (viz. Example 1) may feature *effective domains* $u_i^{-1}(\mathbb{R}) =: dom u_i$ that are proper subsets of $\mathbb{X}$. When X_i is interior to $dom u_i$, no problems emerge. Otherwise, agent i must care that $x_i \in X_i \cap dom u_i$. Such concerns motivate further studies.

It also deserves mention that, in practice, barters and commodity transfers could generate extra cost - or cause some inertia. Yet here, they were implemented with no expense or hesitation. To mitigate this objectionable feature, one could envisage that agent i invokes a *regularized objective*

$$u_i(x) := \max \left\{ \tilde{u}_i(\tilde{x}) - a_i(\tilde{x}, x) \ \mid \tilde{x} \in \mathbb{X} \right\}.$$

[11] Distributed algorithms of block-coordinate projected gradient type are studied in [16], [26] and [25].

[12] Admittedly, perishable and durable goods are traded on different markets. (Think about fresh fish versus fish quotas.) For perishables, one has long seen double auctions, implemented and executed prior to consumption. Similarly, for durables, there are platforms for exchange of property and rental rights.

The underlying $\tilde{u}_i : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ reports his proper revenue whereas $a_i : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ accounts for *adjustment* or *transaction cost*. Presumably, $a_i(\tilde{x}, x) \geq 0$, and $a_i(x, x) = 0$. Quite reasonably, the regularization could also require that $\tilde{x} \in X_i$.

Besides its appeal, regularization often brings an extra bonus: it's apt to smoothen the resulting objective. Specifically, provided $\tilde{u}_i(\tilde{x}) - a_i(\tilde{x}, x)$ be strictly concave in $\tilde{x}$ and differentiable in x, the maximizing $\tilde{x} = \tilde{x}(x)$ is unique, and - by Danskin's envelope theorem [4] - the derived criterion u_i becomes differentiable with

$$u_i'(x) = -\frac{\partial}{\partial x} a_i(\tilde{x}, x).$$

Plainly, an agent who regularizes his objective, appears competent qua optimizer. But this feature does not square with how he was portrayed here. So, although interesting, I have not considered the effects of iterated regularizations. It merits mention though, that presence of merely one (smooth or) regularizing agent, having a barrier function as criterion, largely contributes towards convergence.

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References

1. H. H. Bauschke and J. M. Borwein, On the convergence of von Neumann's alternating projection algorithm for two sets, *Set-Valued Analysis* 1, 183-212 (1993)
2. M. S. Bazaraa, H. D. Sherali and C.M. Shetty, *Nonlinear Programming*, J. Wiley & Sons, New York (1993)
3. G. Dalton, Aboriginal economics in stateless societies: interaction spheres, in J. Erickson and T. Earle (eds.) *Exchange Systems in Pre-History,* Academic Press, New York (1977)
4. J. M. Danskin, *The Theory of Max-Min and its Applications to Weapons Allocations Problems*, Springer Verlag, Berlin (1967)
5. G. B. Danzig and P. Wolfe, The decomposition algorithm for linear programs, *Econometrica* 29, 767-778 (1961)
6. A. M. Feldman, Bilateral trading processes, pairwise optimality, and Pareto optimality, *The Review of Economic Studies* 4, 463-473 (1973)
7. S. D. Flåm and L. Koutsougeras, Private information, transferable utility, and the core, *Economic Theory* 42, 591-609 (2010)
8. S. D. Flåm, On sharing of risks and resources, in S. Reich and A. Zaslavksi (eds.) *Optimization Theory and Related Topics,* American Mathematical Society, series Contemporary Mathematics 568, 53-68 (2012) http://dx.doi.org/10.1090/conm/568/11276
9. S. D. Flåm, Exchanges and measures of risk, *Mathematics and Financial Economics* 5,4, 249-267 (2012)

10. S. D. Flåm, O. Godal and A. Soubeyran, Gradient Differences and Bilateral Barters, *Optimization* 1-20, http://dx.doi.org/10.1080/02331934.2012.679940 (2012)
11. S. D. Flåm, Coupled projects, core imputations, and the CAPM, *J. Mathematical Economics 48, 3, 170-176* (2012)
12. S. D. Flåm and K. Gramstad, Direct exchanges in linear economies, *Int. Game Theory Review* 14 (2012)
13. D. K. Gode and S. Sunder, Allocative efficiency of markets with zero intelligence traders: markets as a partial substitute for individual rationality, *Journal of Political Economy* 101, 119-37 (1993)
14. J-B. Hiriart-Urruty, *Bases, outils et principes pour l'analyse variationelle*, Springer, Berlin (2012)
15. B. S. Mordukhovich and N. M. Nam, *An Easy Path to Convex Analysis and Applications,* Morgan & Claypool Publishers (2014)
16. I. Necoara, Y. Nesterov and F. Glineur, A random coordinate descent method on large-scale optimization problems with linear constraints, Typescript (2015)
17. M. J. Osborne and A. Rubinstein, *Bargaining and Markets*, Academic Press, New York (1990)
18. G. Owen, The core of linear production games, *Mathematical Programming* 9, 358-370 (1975)
19. P. Malo and T. Pennanen, Reduced form modelling of limit order markets, Typescript (2010)
20. R. T. Rockafellar and J-B. Wets, *Variational Analysis*, Springer Verlag, Berlin (1998)
21. A. E. Roth and M. O. Sotomayor, *Two-sided matching: A study in game-theoretic modeling and analysis*, Cambridge Univ. Press, New York (1990)
22. A. Rubinstein and A. Wolinsky, Decentralized trading, strategic behavior and the Walrasian outcome, *The Review of Economic Studies* 57, 63-78 (1990)
23. L. Shapley and M. Shubik, Trade using one commodity as a means of payment, *Journal of Political Economy* 85, 937-968 (1977)
24. V. L. Smith, *Papers in Experimental Economics*, Cambridge University Press, Cambridge (1991)
25. P. Tseng and S. Yun, Block-coordinate gradient descent method for linearly constrained nonsmooth separable optimization, *Journal of Optimization Theory and Applications* 140, 513-535 (2009)
26. L. Xiao and S. Boyd, Optimal scaling of a gradient method for distributed resource allocation, *Journal of Optimization Theory and Applications* 129, 3, 469-488 /2006)

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