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DIRECT EXCHANGE IN LINEAR ECONOMIES.

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Direct Exchange in Linear Economies

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Abstract. Considered here is direct exchange of production allowances or input factors. Motivated by practical modelling and compution, we suppose every owner or user of such items has a linear technology. The issue then is whether competitive market equilibrium can be reached merely via iterated bilateral barters. This paper provides positive and constructive answers.

1. Introduction

Consider an economy in which agents own specific production factors and linear technologies. Those factors and the resulting payoffs are regarded as perfectly transferable. In such a setting, this paper considers whether market equilibrium could be reached merely via direct two-sided deals.

What we have in mind is bilateral exchange of natural resources or user-rights to such. Important instances comprise transfers of fish quotas, production allowances, pollution permits, or rights to water usage. Also fitting is trade of insurance policies and contingent claims [6].

Many natural resources and production externalities are not traded - or their markets are not well developed. Nonetheless, one can often witness direct transfer of user permits, rental rights, or property shares.[1] Typically, and quite naturally, such transactions are facilitated by side payments. That is, money oils the exchange mechanism. Reflecting on this feature, we ask: *Can the agents reach equilibrium merely via iterated bilateral barters? If so, how?*

Feldman (1973) already studied these issues in a pioneering paper. He required that each deal yield a *core outcome* for the two parties who undertake exchange.[2] Our approach - while decidedly agent-based and computational - is more inspired by behavioral and experimental economics [27]. So, it takes another tack: no joint

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[1]So-called unitization of oil fields are cases in point.
[2]Other studies include [13], [15], [17], [23] and [26].

optimization is ever undertaken. Agents merely adapt, step by step, each move being somewhat moderate and myopic.

Along that line, the paper supplements and adds to closely related, on-going studies [7], [8]. Its novelties - and its many constructive features - come by specializing to linear economies. Within that important, quite applicable setting, the paper provides positive answers to the above questions. Decentralized and direct two-sided deals may indeed suffice to generate market clearing prices. Under weak assumptions, no coordination is ever needed. This result speaks for the stability of equilibrium in stationary, competitive economies.

The paper is planned as follows. Section 2 deals with existence of equilibrium. Section 3 models exchange as direct, between only two parties at a time, and iteratively driven merely by differences in their margins. Section 4 details the manner of exchange - presented there with the flavor of an algorithm. Section 5 proves convergence. Some examples are found in Section 6.

2. ECONOMIC EQUILIBRIUM

This section formalizes the linear economy, defines the concept of transferable-utility competitive equilibrium - and studies existence of such outcomes.

Agent i owns *endowment* e_i of production factors, and he gets monetary payoff $\pi_i(x_i)$ upon using input x_i. Both bundles e_i, x_i are construed as vectors in a real linear space $\mathbb{X}$.

Our motivation is partly computational, and the orientation is agent-based [28]. Accordingly, there is finite ensemble I of adaptive agents, and a finite list C of commodities. Let $\mathbb{X} = \mathbb{R}^C$ be the set of all functions $x : C \to \mathbb{R}$. Writing such commodity bundles in the form $x = [x_c]$, we equip $\mathbb{X}$ with ordinary inner product $x \cdot \hat{x} := \sum_{c \in C} x_c \hat{x}_c$, associated norm $\|x\| = (x \cdot x)^{1/2}$, and standard vector order $x \geq \hat{x} \Leftrightarrow x_c \geq \hat{x}_c \; \forall c \in C$.[3]

Further, also for the sake of computation and practical modelling, suppose all production payoffs derive from *linear programming* [14]. Specifically, let

$$\pi_i(x_i) := \sup \left\{ y_i^* \cdot y_i \mid x_i \geq A_i y_i \; \& \; y_i \in Y_i \right\}. \tag{1}$$

The interpretation of (1) goes as follows. Agent i must choose a production plan $y_i \in Y_i$. The set Y_i, which accounts for his activity constraints, is presumed *polyhedral*.[4] It is part of a linear space $\mathbb{Y}_i$, having finite dimension and inner product $y_i^* \cdot y_i$. The vector $y_i^* \in \mathbb{Y}_i$ reflects agent i's linear objective. A matrix or linear mapping $A_i : \mathbb{Y}_i \to \mathbb{X}$ represents his technology. When operative, that technology consumes various production factors - of which instance (1) lets bundle $x_i \in \mathbb{X}$ be available. The subsequent discussion also fits the standard version of (1) in which $x_i = A_i y_i$.

We remark that agent model (1) is important in practice - and most tractable for computation. It can fit directly to a productive enterprise or serve as a good approximation to what goes on there.

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[3]In asset or insurance markets, $x_c \in \mathbb{R}$ records a monetary *claim*, valid only in *contingency $c \in C$*.

[4]A set which equals the intersection of finitely many closed half-spaces is declared *polyhedral*.

Clearly, to make *economic sense*, input x_i must yield finite value $\pi_i(x_i)$ in (1).[5] So, by assumption, $\pi_i < +\infty$ everywhere - and, of course, $\pi_i > -\infty$ somewhere. Thus, as a matter of hypothesis, agent i's *effective domain*

$$dom\pi_i := \{x_i \in \mathbb{X} \mid \text{the optimal value } \pi_i(x_i) \text{ is finite}\}$$

is non-empty. Further, to make *physical sense*, x_i should belong to some prescribed subset $\mathbb{X}_i \subseteq \mathbb{X}$.[6] Accordingly, agent i must always care that his input x_i be member of

$$X_i := \mathbb{X}_i \cap dom\pi_i.$$

Henceforth, by assumption, X_i *is closed convex, and* π_i *is finite-valued near any* $x_i \in X_i$. The latter qualification amounts to require that

$$X_i \subset int(dom\pi_i). \tag{2}$$

To postpone queries about feasibility, let the indicator $\delta_i : \mathbb{X} \to \{0, -\infty\}$ take the value $\delta_i(x_i) = 0$ if $x_i \in X_i$, and $-\infty$ elsewhere. This extreme penalty function simply serves to report violation of implicit constraints. Using this function, the essential objective of agent i reads

$$u_i := \pi_i + \delta_i. \tag{3}$$

In autarky he could, at most, achieve payoff $u_i(e_i)$. But voluntary exchanges often bring widespread improvements. Notably, a competitive market might make marvels. In the present context, featuring perfectly transferable payoffs and production factors, price-supported equilibrium becomes especially tractable:

Definition (Exchange market equilibrium). *The input profile* $(x_i) \in \mathbb{X}^I$ *alongside a price* $p \in \mathbb{X}$ *is declared a competitive* **equilibrium** *iff* $\sum_{i \in I} x_i = e_I := \sum_{i \in I} e_i$, *and*

$$u_i(x_i) + p \cdot (e_i - x_i) \geq u_i(\chi_i) + p \cdot (e_i - \chi_i) \quad \text{for each} \quad \chi_i \in \mathbb{X} \quad \text{and} \quad i \in I. \tag{4}$$

Thus, the solution concept requires that demand equals supply, and that everybody maximizes his direct payoff $u_i(x_i)$ plus price-taking revenue $p \cdot (e_i - x_i)$ from trading resources.[7]

Granted format (1) and perfect transferability, competitive equilibria exist under weak and natural assumptions. For a statement to that fact, call a subset X of a real vector space *conical* iff $(\underline{r}, +\infty)X \subseteq X$ for some real $\underline{r} \geq 0$.

Proposition 2.1 (Existence of competitive equilibrium). *Suppose* (1) *holds, and that the optimal value*

$$u_I(x) := \sup \left\{ \sum_{i \in I} u_i(x_i) \,\middle|\, \sum_{i \in I} x_i = x \right\} \tag{5}$$

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[5]As customary, if x_i renders (1) infeasible, posit $\pi_i(x_i) = -\infty$.
[6]For an important and natural example, take $\mathbb{X}_i = \mathbb{X}_+ = \mathbb{R}_+^C$.
[7]It is, of course, tacitly assumed that each $u_i(x_i)$ be finite in equilibrium.

is finite and attained at the aggregate endowment $x = e_I$. Then there exists an equilibrium (x_i). When X_i and Y_i are conical, $u_i(\chi_i) \leq p \cdot \chi_i$ for each $\chi_i \in \{\pi_i \geq 0\}$. If moreover, X_i and Y_i are cones, then $u_i(x_i) = p \cdot x_i$.

Proof. Inserting (1) in (5), and letting $x = e_I$, yields the extended linear program:

$$u_I(x) \;=\; \sup\left\{\sum_{i \in I} y_i^* \cdot y_i \;\middle|\; x_i \geq A_i y_i,\, y_i \in Y_i \;\&\; \sum_{i \in I} x_i = x\right\}$$

$$\;=\; \sup\left\{\sum_{i \in I} y_i^* \cdot y_i \;\middle|\; x \geq \sum_{i \in I} A_i y_i,\, y_i \in Y_i\right\}.$$

By assumption, this fully coordinated program has finite value - whence at least one optimal solution $(x_i) \in \mathbb{X}^I$. Consequently, the corresponding *dual program* [14] admits an optimal solution in which a multiplier $p \in \mathbb{X}$ associates to the coupling constraint $\sum_{i \in I} x_i = \sum_{i \in I} e_i$. The reduced Lagrangian

$$L = \sum_{i \in I} \{u_i(\chi_i) + p \cdot (e_i - \chi_i)\}$$

is then maximal in (χ_i) at (x_i). Such maximality is equivalent to (4).

Finally, if X_i, Y_i are conical, $\pi_i(r\chi_i) \geq r\pi_i(\chi_i) \geq 0$ when r is sufficiently large and $\chi_i \in X_i \cap \{\pi_i \geq 0\}$. If one inequality is strict, then $lim_{r \to +\infty} \pi_i(r\chi_i) = +\infty$ - a contradiction. Hence, any equilibrium price p must yield $u_i(\chi_i) \leq p \cdot \chi_i$ for all $\chi_i \in \{\pi_i \geq 0\}$. $\square$

Proposition 2.1 generalizes that of Owen (1975), concerned with *core solutions* to equal-objective, equal-technology, linear production games with transferable utility. Here, since our focus is on competitive outcomes, by the *first fundamental welfare theorem*, equilibria automatically belong to the core [29].

While existence easily obtains, uniqueness can hardly be guaranteed. For example, suppose agents i, j have identical programs (1), and $\mathbb{X}_i = \mathbb{X}_j$. Then, if an equilibrium allocation has $x_i \neq x_j$, these two parties could equally well swap their inputs.[8] Also, if the overall dual program, mentioned above, has several optimal solutions, then equilibrium price vectors are just as many.

Linear models of exchange have a long and fascinating history.[9] Their importance make us inquire here about the attainability of market balance - and the emergence of equilibrating prices. *How might non-coordinated, scantly informed parties come to implement equilibrium by themselves?* Intuition suggests that iterated bilateral barters might suffice. The next section explores this idea.

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[8] For a trivial illustration, let $I = \{1, 2\}$, $\pi_i(x_i) = x_i$, $X_i = [0, 1]$, and $e_i = 1/2$. Then each feasible allocation is an equilibrium.

[9] See Bray (1922), Remak (1929), Frisch (1934), Gale (1960).

3. DIRECT EXCHANGE

This section spells out the main idea behind our modelling of bilateral barters, illustrated by a simple example. Also included here are a few facts on shadow prices and their coincidence in equilibrium.

The recurrent trade episode goes as follows. Suppose two agents i, j meet, then holding factor bundles $x_i \in X_i$, $x_j \in X_j$ respectively. If they agree on a suitable direction $\Delta \in \mathbb{X}$ of input transfer to i from j, and use a step-size $\sigma > 0$ along that direction, their tentatively updated holdings become

$$x_i + \sigma\Delta \quad \text{and} \quad x_j - \sigma\Delta.$$

On choice of transfer direction: the smooth case. *What direction Δ might appear attractive?* For the sake of simple argument, first suppose u_i, u_j are differentiable at x_i, x_j respectively, with gradients $g_i = u_i'(x_i)$, and $g_j = u_j'(x_j)$ there. The gradient difference

$$\Delta := g_i - g_j \tag{6}$$

then seems a good first proposal. The reason is simple: using small enough step-size $\sigma > 0$, if $\Delta \neq 0$, the value added

$$\Delta u_{ij}(\sigma) = u_i(x_i + \sigma\Delta) + u_j(x_j - \sigma\Delta) - u_i(x_i) - u_j(x_j)$$

becomes strictly positive. Indeed, $\Delta u_{ij}'(0) = \left[u_i'(x_i) - u_j'(x_j) \right] \cdot \Delta = \|\Delta\|^2 > 0$.

However, the proposed and promising direction Δ - or the associated step-size σ - might cause some update to become infeasible. That is, the two trading parties must care that $x_i + \sigma\Delta \in X_i$ and $x_j - \sigma\Delta \in X_j$. In this regard, to disentangle different concerns, we first look at their choice of some appropriate direction. For precise discussion and statement, let, in general,

$$D(X, x) := \{ r(\chi - x) \mid r \geq 0 \text{ and } \chi \in X \} = \mathbb{R}_+(X - x)$$

denote the *cone of feasible directions* of a subset $X \subseteq \mathbb{X}$ at $x \in X$. As one might expect, the candidate direction Δ must, when necessary, be bent onto the convex cone $D_i x_i := D(X_i, x_i)$. Here, given the polyhedral nature of all sets, we *assume that any such cone $D_i x_i$ is closed.*

Quite similarly, looking at i's interlocutor and counterpart j, his opposite direction $-\Delta$ must, maybe after mandatory bending, reside in $D_j x_j$. In short, to make the chosen direction feasible for both, let

$$d = P_{ij}\Delta \tag{7}$$

denote the *orthogonal projection* of Δ onto the convex cone

$$D_{ij}(x_i, x_j) := D_i x_i \cap -D_j x_j,$$

assumed closed.[10]   In view of polyhedral instances, we tacitly reckon that agents i, j experience little difficulties in executing projection P_{ij}; see Section 6.

Next comes a simple illustration:

Example (*An elementary linear exchange economy*). Let $C = \{1, 2, 3\}$, and consider three agents $i = 1, 2, 3$, boxed next:

$$
\begin{array}{ll}
\pi_1(x_1) = & 3x_{11} + 2x_{12} + 1x_{13}, \quad e_1 = (0, 1, 0), \\
\pi_2(x_2) = & 2x_{21} + 1x_{22} + 3x_{23}, \quad e_2 = (1, 0, 0), \\
\pi_3(x_3) = & 1x_{31} + 3x_{32} + 2x_{33}, \quad e_3 = (0, 0, 1).
\end{array}
$$

For lighter notation, $x_{ic} := x_i(c)$. This instance fits, of course, the standard version of (1): Let A_i be the $C \times C$ identity matrix, let $x_i = A_i y_i$, posit $\mathbb{Y}_i = \mathbb{R}^C$ with ordinary inner product, and choose linear objectives defined by vectors $y_1^* = (3, 2, 1)$, $y_2^* = (2, 1, 3)$, $y_3^* = (1, 3, 2)$.

Plainly, π_i is finite-valued everywhere. Let $X_i = \mathbb{X}_+ = \mathbb{R}_+^C$. Write simply D for the cone D_i, and P for the projection P_{ij}. As it turns out, exchange can proceed with constant step-size $\sigma = 1$ until swift convergence - as follows:

In round 1 suppose agents $1, 2$ trade, starting from initial allocation $[x_i^0] = [e_i]$. Since $Dx_1^0 = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and $Dx_2^0 = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$, we get $Dx_1^0 \cap -Dx_2^0 = \mathbb{R}_+ \times \mathbb{R}_- \times \{0\}$ and $\Delta = \pi_1' - \pi_2' = (1, 1, -2)$. So, the two active traders may apply projected direction $d = P\Delta = (1, 0, 0)$. Using step-size $\sigma = 1$, the updated allocation becomes $[x_i^1] = [(1, 1, 0), (0, 0, 0), (0, 0, 1)]$.

In round 2 let agents $1, 3$ trade. Since $Dx_1^1 \cap -Dx_3^1 = \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+$ and $\Delta = \pi_1' - \pi_3' = (2, -1, -1)$, let $d = P\Delta = (0, -1, 0)$ to produce a novel allocation $[x_i^2] = [(1, 0, 0), (0, 0, 0), (0, 1, 1)]$.

Finally, in round 3 suppose agents $2, 3$ trade to arrive at equilibrium allocation $[x_i] = [x_i^3] = [(1, 0, 0), (0, 0, 1), (0, 1, 0)]$, and associated equilibrium price $p = (3, 3, 3)$.

In each round, the two traders obtain an increment

$$\Delta\pi_{ij}(\sigma) := \pi_i(x_i + \sigma d) + \pi_j(x_j - \sigma d) - \pi_i(x_i) - \pi_j(x_j) \tag{8}$$

in joint payoff which equals $\Delta\pi_{ij}(1) = 1$. The "winner" gets additional payoff 3, and the "looser" gives up 2. Accordingly, if the first pays compensation 2.5 to the latter, both can pocket 0.5. $\Diamond$

More on choice of transfer direction: the case of non-smooth objectives. Our arguments hitherto for choosing direction (7) hinged on u_i, u_j being differentiable. Alas, quite often, they are not. The optimal value function π_i, as defined in (1), need not have a classical gradient at $x_i \in X_i$.[11]   More seriously, the extreme

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[10]We note that projection P_{ij} is positively homogeneous. So, when well defined, only the unit vector $d/\|d\|$ counts, along which i receives a transfer.

[11]As is well known, π_i is (continuously) differentiable at x_i iff $\partial\pi_i(x_i)$ reduces to a singleton. By Rademacher's theorem this happens on a dense subset of X_i.

penalty function δ_i is dramatically non-smooth at the boundary of X_i. So, in the sequel, we must contend with use of *generalized gradients* [21], [22].

Recall that $g \in \mathbb{X}$ is called a *supergradient* of a function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ at $x \in \mathbb{X}$, and we write $g \in \partial f(x)$, iff $f(x)$ is finite, and

$$f(\chi) \leq f(x) + g \cdot (\chi - x) \quad \text{for all} \ \chi \in \mathbb{X}.$$

We presume that traders will not spoil the simplicity of procedures (6) and (7). Yet, dispensing now with differentiability, there remains a question: *what supergradients might, via their differences, drive trade?* To address this issue, we record next how optimal values like (1) are "differentiated." The following, well known result derives from Danskin's envelope theorem [3] and linear programming duality [14]. For simplicity in statement, let

$$\sigma_Y(y^*) := \sup \{y^* \cdot y \mid y \in Y\}$$

be the *support function* of some subset Y of a Euclidean space $\mathbb{Y}$. As usual, when $A : \mathbb{Y} \to \mathbb{X}$ is linear, $A^T : \mathbb{X} \to \mathbb{Y}$ denotes its transpose operator. While A maps "activity" $y \in \mathbb{Y}$ into "resource consumption" $Ay = x \in \mathbb{X}$, the transpose A^T sends linear resource prices x^* on $\mathbb{X}$ into corresponding activity prices y^* on $\mathbb{Y}$.

Proposition 3.1 (Supergradients of linear programs). *Suppose, as in* (1), *that*

$$\pi(x) := \sup \{y^* \cdot y \mid x \geq Ay \ \& \ y \in Y\}$$

emerges as the optimal value of a linear program in which $Y \subseteq \mathbb{Y}$ is polyhedral. Then, the function $\pi : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$, so defined, becomes concave, and its superdifferential $\partial\pi(x)$ is non-empty whenever $\pi(x)$ is finite. To wit, $x^* \in \partial\pi(x)$ iff x^* solves the corresponding dual program with equal value, namely:

$$\pi(x) = \inf \left\{x^* \cdot x + \sigma_Y(y^* - A^T x^*) \mid x^* \geq 0\right\}. \quad \square$$

This proposition applies of course to each program (1). Further, for differentiation of essential objective u_i (3), recall the concept of *normal cone*

$$N(X, x) := \{x^* \in \mathbb{X} \mid x^* \cdot (\chi - x) \leq 0 \text{ for all } \chi \in X\}$$

to a subset $X \subseteq \mathbb{X}$ at $x \in X$. Now, in terms of $N_i(x_i) := N(X_i, x_i)$, we have $\partial\delta_i(x) = -N_i(x_i)$ so that

$$\partial u_i(x_i) = \partial\pi_i(x_i) - N_i(x_i).$$

Any $p_i \in \partial u_i(x_i)$ is commonly called a *shadow price*. To conclude this section, we note straightforwardly that equilibrium prevails when the agents' shadow prices all coincide:

Proposition 3.2 (Equilibrium and a common shadow price). *If* $\sum_{i \in I} x_i = \sum_{i \in I} e_i$ *and*

$$p \in \cap_{i \in I} \partial u_i(x_i), \tag{9}$$

then allocation (x_i) *and price p constitute an equilibrium. Conversely, if* $(x_i), p$ *is an equilibrium, (9) holds, and* $p \in \partial u_I(e_I)$. $\square$

Equilibrium might be construed as "the goal" of some central planner or imaginary auctioneer. None of the many narratives that go along with such perspectives fit here [24]. Our interest is rather with non-coordinated enterprises - and with agents' behavior out of equilibrium. A candidate mode of such behavior is described next.

4. BILATERAL BARTERS

Only for argument and discussion, we find it expedient to let the exchange process unfold very much like an *algorithm,* fictitious or real, but affected by a *protocol* that decides who will trade next with whom.

Repeated bilateral barters construed as an ascent algorithm:
- *Start* with some allocation $i \mapsto x_i \in X_i$ such that $\sum_{i \in I} x_i = \sum_{i \in I} e_i$.
- *Invoke the protocol* to activate a novel agent pair $i, j \in I$.
- *Pick supergradients* $g_i \in \partial u_i(x_i)$ and $g_j \in \partial u_j(x_j)$. If $d := P_{ij}(g_i - g_j) = 0$, *invoke the protocol* anew.
- *Stop* when all $d = 0$.
- Otherwise, *choose a feasible step-size* $\sigma = \sigma_{ij} > 0$.
- If in (8) $\Delta \pi_{ij}(\sigma) \geq 0$, *update holdings*:

$$x_i \leftarrow x_i + \sigma d \in X_i \quad \text{and} \quad x_j \leftarrow x_j - \sigma d \in X_j. \tag{10}$$

- *Continue* to *invoke the protocol* until convergence.

Note that $\sigma > 0$ and $d \neq 0$ whenever (10) is executed. We say that i, j then undertake a *real trade*.

Clearly, the algorithm maintains $\sum_{i \in I} x_i = e_I$ and $x_i \in X_i$ throughout. The above schematic outline glosses, of course, over much detail, institutional and narrative. It reflects though, that autonomous agents, who operate within reasonable legal frames, often find improving allocations.

Upon viewing exchange as the implementation of a (pre)programmed procedure, one may wonder about its convergence rate. We shall make no claims in this regard. Our only - and more modest - concern is with asymptotic stability: *will holdings cluster to efficient allocations?* More to the point: *will a common shadow price eventually emerge?*

These questions invite queries about *stopping, step-sizes, protocol,* and *convergence.* We address the first three issues next, deferring discussion of convergence to the subsequent section.

The stopping criterion is idealized and imprecise here. In practice, exchange terminates when
* either all projected gradient differences $d = P_{ij}(g_i - g_j)$ are so small (in norm) as to pass unnoticed, or if
* the relative increment $\Delta \sum_i u_i / \sum_i u_i$ is negligible over some suitably long period.

For a more theoretical perspective on stopping, we state the following result, easily proved - and best understood as a special instance of Proposition 3.2:

Proposition 4.1 (A common shadow price blocks bilateral barter). *Suppose agents i, j, while holding $x_i \in X_i, x_j \in X_j$, meet to contemplate a bilateral exchange. Then, at any optimal solution*

$$(x_i^{+1}, x_j^{+1}) \in \arg\max \left\{ u_i(\tilde{x}_i) + u_j(\tilde{x}_j) \mid \tilde{x}_i + \tilde{x}_j = x_i + x_j \right\}$$

they see a common shadow price $p \in \partial u_i(x_i^{+1}) \cap \partial u_j(x_j^{+1})$.
Conversely, if they already see one such price at $x_i \in X_i$, $x_j \in X_j$, then $(x_i, x_j) = (x_i^{+1}, x_j^{+1})$ solves the above $\arg\max$ inclusion. $\square$

Proposition 4.1 tells that i, j should *not* barter if they see a common shadow price. But otherwise, *if $\partial u_i(x_i)$ does not intersect $\partial u_j(x_j)$, what feasible direction $d \in D_{ij}(x_i, x_j)$ might be best?* The method of steepest ascent will guide us here:

Definition (Maximal slope). *Suppose $x_i \in X_i$ and $x_j \in X_j$. Then, by the **maximal slope of joint improvement** for agents i, j is meant*

$$\mathfrak{S}_{ij}(x_i, x_j) := \max \left\{ \pi_i'(x_i; \mathbf{d}) + \pi_j'(x_j; -\mathbf{d}) \mid \mathbf{d} \in D_{ij}(x_i, x_j) \ \& \ \|\mathbf{d}\| \leq 1 \right\}. \quad (11)$$

The chosen direction should emerge as

$$d = P_{ij}(g_i - g_j) \quad \text{with} \quad g_i \in \partial u_i(x_i) \quad \text{and} \quad g_j \in \partial u_j(x_j). \quad (12)$$

As said, *when (10) & (12) hold with $\sigma > 0$, $d \neq 0$, and $\Delta\pi_{ij} \geq 0$, agents i, j make a **real trade**.*
One wonders: In (12) what supergradients g_i, g_j make $d = P_{ij}(g_i - g_j)$ line up with a maximizing $\mathbf{d}$ in (11)? The next result settles this question. For its statement, given any two non-empty closed convex subsets $\mathcal{C}_i, \mathcal{C}_j \subseteq \mathbb{X}$, naturally define their proximity, gap, or minimal *distance* by

$$dist\,[\mathcal{C}_i, \mathcal{C}_j] := \inf \left\{ \|c_i - c_j\| \mid c_i \in \mathcal{C}_i, c_j \in \mathcal{C}_j \right\}.$$

Proposition 4.2 (A best common direction for bilateral transfer). *When distinct agents, i, j, hold feasible bundles $x_i \in X_i$ and $x_j \in X_j$ respectively, their maximal*

slope of joint improvement equals

$$\begin{aligned}
\mathfrak{S}_{ij}(x_i, x_j) &= \min\left\{\|P_{ij}\left[g_i - g_j\right]\| \;\middle|\; g_i \in \partial\pi_i(x_i),\; g_j \in \partial\pi_j(x_j)\right\} \\
&= \min\left\{\|g_i - g_j\| \;\middle|\; g_i \in \partial u_i(x_i),\; g_j \in \partial u_j(x_j)\right\} \\
&= dist\left[\partial u_i(x_i), \partial u_j(x_j)\right].
\end{aligned}$$

It follows that $\mathfrak{S}_{ij}$ is lower semicontinuous on $X_i \times X_j$.

Proof (preferably not included in the final version). Recall that a concave function f which is finite near x, has a non-empty compact convex *superdifferential* $\partial f(x)$ and a *directional derivative*

$$f'(x; \mathbf{d}) := \lim_{r \to 0^+} \frac{f(x + r\mathbf{d}) - f(x)}{r} = \min\left\{x^* \cdot \mathbf{d} \;\middle|\; x^* \in \partial f(x)\right\}. \qquad (13)$$

Qualification (2) ensures that $\partial\pi_i(x_i)$ is a non-empty compact convex set whenever $x_i \in X_i$. For this reason, letting B denote the closed unit ball, it obtains $\mathfrak{S}_{ij}(x_i, x_j)$

$$= \max_{\mathbf{d}} \min_{g_i, g_j} \left\{(g_i - g_j) \cdot \mathbf{d} \;\middle|\; g_i \in \partial\pi_i(x_i),\; g_j \in \partial\pi_j(x_j),\; \mathbf{d} \in D_{ij}(x_i, x_j) \cap B\right\}$$

$$= \min_{g_i, g_j} \max_{\mathbf{d}} \left\{(g_i - g_j) \cdot \mathbf{d} \;\middle|\; g_i \in \partial\pi_i(x_i),\; g_j \in \partial\pi_j(x_j),\; \mathbf{d} \in D_{ij}(x_i, x_j) \cap B\right\}$$

$$= \min\left\{\|P_{ij}\left[g_i - g_j\right]\| \;\middle|\; g_i \in \partial\pi_i(x_i),\; g_j \in \partial\pi_j(x_j)\right\}.$$

In the preceding string, the first equality simply used formula (13) and the definition of the maximal slope. Since all intervening sets are non-empty compact convex, the second equality follows from the von Neumann min-max theorem. The third equality derives from the Cauchy-Schwartz inequality, using the Moreau decomposition of $\Delta := g_i - g_j$ with respect to the convex cone $D_{ij}(x_i, x_j)$, presumed closed. That is, Δ decomposes uniquely into a sum $d + n$ where $d = P_{ij}\Delta$ is feasible and n is normal to $D_{ij}(x_i, x_j)$, meaning $n \cdot D_{ij}(x_i, x_j) \leq 0$.

This proves the first equality in the proposition. For the last two, note that

$$\mathcal{C}_i := \partial u_i(x_i) = \partial\pi_i(x_i) - N_i(x_i) \quad \text{and} \quad \mathcal{C}_j := \partial u_j(x_j) = \partial\pi_j(x_j) - N_j(x_j) \qquad (14)$$

are non-empty closed convex subsets of $\mathbb{X}$ (because $\partial\pi_i(x_i), \partial\pi_j(x_j)$ are compact convex). Also note that $D_i x_i$ equals the *dual cone* $\{x^* \mid x^* \cdot N_i(x_i) \leq 0\}$ of $N_i(x_i)$ - and likewise for $D_j x_j$. It follows therefore from that $dist\left[\mathcal{C}_i, \mathcal{C}_j\right] = dist\left[\partial u_i(x_i), \partial u_j(x_j)\right]$

$$= \inf\left\{\|g_i - g_j\| \;\middle|\; g_i \in \partial u_i(x_i),\; g_j \in \partial u_j(x_j)\right\}$$

$$= \min\left\{\|g_i - g_j\| \;\middle|\; g_i \in \partial u_i(x_i),\; g_j \in \partial u_j(x_j)\right\}$$

$$= \min\left\{\|P_{ij}\left[g_i - g_j\right]\| \;\middle|\; g_i \in \partial\pi_i(x_i),\; g_j \in \partial\pi_j(x_j)\right\}.$$

Finally the lower semicontinuty of $\mathfrak{S}_{ij}$ at any $(x_i, x_j) \in X_i \times X_j$ follows from the outer continuity of the correspondences in (14); see [22]. This completes the proof. $\square$

In hindsight, and not surprisingly, when the norm of $d = P_{ij}[g_i - g_j] \neq 0$ realizes $\mathfrak{S}_{ij}(x_i, x_j)$, the maximizing unit vector in (11) is $\mathbf{d} = d/\|d\|$.

Proposition 4.2 tells that traders who implement a maximal slope can proceed as though non-smoothness creates no hurdles. Yet, securing maximal ascent of joint payoff can overstretch their competence - or require too much computational effort on their part. So, why could they not contend with a fraction of the said slope? Along that line we make a standing assumption.

On choice of unit direction: *If $\mathfrak{S}_{ij}(x_i, x_j) > 0$ and agents i, j really trade, they use a unit-vector direction $\mathbf{d} \in D_{ij}(x_i, x_j)$ such that*

$$\varphi_{ij} \mathfrak{S}_{ij}(x_i, x_j) \leq \pi'_i(x_i; \mathbf{d}) + \pi'_j(x_j; -\mathbf{d}).$$

Here $\varphi_{ij} \in (0, 1)$ is a fixed fraction of the maximal slope. In other words, when $d = P_{ij}(g_i - g_j) = \mathbf{d}\|d\| \neq 0$,

$$\varphi_{ij} \mathfrak{S}_{ij}(x_i, x_j) \|d\| \leq \pi'_i(x_i; d) + \pi'_j(x_j; -d). \tag{15}$$

For brevity, *call any vector $d \in D_{ij}(x_i, x_j)$ that satisfies (15) an **approximate direction**.* As we shall see, use of such directions allows us to bracket the *payoff increment* (8).

Proposition 4.3 (Bracketing the payoff increment). *Granted $\mathfrak{S}_{ij}(x_i, x_j) > 0$ suppose supergradients $g_i \in \partial u_i(x_i)$, $g_j \in \partial u_j(x_j)$ yield an approximate direction $d = P_{ij}(g_i - g_j) \neq 0$. Then, for small enough step-size $\sigma > 0$,*

$$\begin{aligned} \varphi_{ij}^2 \sigma \mathfrak{S}_{ij}(x_i, x_j) \|d\| &\leq \Delta\pi_{ij}(\sigma) \tag{16} \\ &\leq \sigma \min\left\{\|P_{ij}(g_i - g_j)\|^2, \mathfrak{S}_{ij}(x_i, x_j) \|d\|\right\} \\ &= \sigma \mathfrak{S}_{ij}(x_i, x_j) \|P_{ij}(g_i - g_j)\|. \end{aligned}$$

Proof (preferably not included in the final version). Because both functions u_i, u_j are concave, it holds for *any* supergradients $g_i \in \partial u_i(x_i)$, $g_j \in \partial u_j(x_j)$, *any* direction d, and *any* step-size σ, that

$$u_i(x_i^{+1}) \leq u_i(x_i) + \sigma g_i \cdot d \quad \text{and} \quad u_j(x_j^{+1}) \leq u_j(x_j) - \sigma g_j \cdot d.$$

So, upon adding these two inequalities, and inserting $d = P_{ij}(g_i - g_j)$, yields (even when $\mathfrak{S}_{ij}(x_i, x_j) = 0$) that

$$\Delta u_{ij}(\sigma) \leq \sigma(g_i - g_j) \cdot d = \sigma \|P_{ij}(g_i - g_j)\|^2. \tag{17}$$

Here, for the last inequality, we used the orthogonal decomposition

$$g_i - g_j = P_{ij}(g_i - g_j) + n = d + n$$

where $n \cdot d = 0$. Clearly, $\Delta u_{ij}(\sigma) = \Delta\pi_{ij}(\sigma)$ when $x_i, x_i + \sigma d \in X_i$ and $x_j, x_j - \sigma d \in X_j$. Note that $\sigma \mapsto \Delta\pi_{ij}(\sigma)$ is concave for small enough σ. Further, since the derivative $\Delta\pi'_{ij}$ is positively homogeneous,

$$\Delta\pi_{ij}(\sigma) \leq \sigma\Delta\pi'_{ij}(0) \leq \sigma\mathfrak{S}_{ij}(x_i, x_j) \, \|d\| \, .$$

Combining this overestimate with (17) gives the right hand inequality in (16). The last equality there derives from Proposition 4.2, saying that $\mathfrak{S}_{ij}(x_i, x_j) \leq \|P_{ij}(g_i - g_j)\|$.

Finally, for the left hand inequality in (16), when $\mathfrak{S}_{ij}(x_i, x_j) > 0$, and direction d is approximate, it holds

$$0 < \varphi_{ij}\mathfrak{S}_{ij}(x_i, x_j) \, \|d\| \leq \Delta\pi'_{ij}(0) = \lim_{\sigma \to 0^+} \frac{\Delta\pi_{ij}(\sigma) - \Delta\pi_{ij}(0)}{\sigma}.$$

Since $\Delta\pi_{ij}(0) = 0$, the last inequality implies $\varphi^2_{ij}\sigma\mathfrak{S}_{ij}(x_i, x_j) \, \|d\| \leq \Delta\pi_{ij}(\sigma)$ for small enough $\sigma > 0$. $\square$

This completes our discussion of feasible directions, approximate or not. It is time now to consider choice of step-sizes. While trade is underway, no agent can step outside his feasible domain. Yet, when prospects for joint improvement are promising, the interlocutors ought opt for a non-negligible transfer - that is, for *real trade*. In that regard we make a standing assumption:

On step-sizes: *For any chosen direction $d \in D_{ij}(x_i, x_j)$ agents i, j agree on a step-size $\sigma = \sigma_{ij} \geq 0$ such that $x_i + \sigma d \in X_i$ and $x_j - \sigma d \in X_j$. Most important, we suppose that **step-sizes dwindle***

$$\sigma^k \to 0, \tag{18}$$

but not too fast: *For any subsequence K, all along which some agent pair i, j really trade, it holds*

$$\lim_{k \in K}(x_i^k, x_j^k) = (x_i, x_j), \quad \sigma_{ij}^k > 0 \ \ \& \ \ \liminf_{k \in K} \mathfrak{S}_{ij}(x_i^k, x_j^k) > 0 \Rightarrow \sum_{k \in K} \sigma_{ij}^k = +\infty. \tag{19}$$

We think these conditions are reasonable. The ultimate vanishing of step-sizes reflects market maturation and reduced volatility in agents' holdings. Yet, if agents i, j indeed trade along a convergent sequence, for which $\mathfrak{S}_{ij} > 0$ in the limit, their step-sizes at those stages should form a divergent series. These assumptions, both on asymptotics, offers agents great freedom. In particular, they may apply maximal steps for a long while. And finite-time convergence is not precluded.

The protocol serves to regulate encounters. The recurrent issue is: *who meets next*

whom? There might be room for random pairing, deliberate search, asynchronous or parallel matching - and different affinities among agents. Broadly, what imports is that each agent pair be activated repeatedly. We shall not single out any specific protocol. But we do assume that agents meet in almost cyclical manner:

Encounters are quasi-cyclic: *for some finite lapse l of stages each pair i, j meets at least once during every interval of length l.*

5.   On Convergence

This section shows, under weak assumptions, that each accumulation point of the sequence $k \mapsto \mathbf{x}^k := (x_i^k)$ is an efficient allocation. Henceforth naturally suppose the set

$$\mathbf{A} = \left\{ \mathbf{x} = (x_i) \;\middle|\; x_i \in X_i, \; \sum_{i \in I} x_i = e_I \right\}$$

of feasible allocations is bounded. Part of the argument is that iterated exchange completely exhausts all possibilities for bilateral improvement. That is, we shall show that the sequence $\{\mathbf{x}^k\}$ clusters to the set

$$\mathbf{C} := \{\mathbf{x} = (x_i) \in \mathbf{A} \mid \text{all } \mathfrak{S}_{ij}(x_i, x_j) = 0\}$$

composed of what we call *complete-trade allocations.* Only thereafter shall we add an assumption which ensures the coincidence of shadow prices.

Proposition 5.1 (On exhaustion of two-sided trade options)*. Each subsequential limit point* $\mathbf{x} = (x_i)$ *of the sequence* $\{\mathbf{x}^k\}$ *belongs to the set* $\mathbf{C}$ *of complete-trade allocations.*

Proof. Let $\mathcal{L}$ denote the set of all limit points of $\{\mathbf{x}^k\}$. At least one such point exists because each $\mathbf{x}^k$ belongs to the bounded set $\mathbf{A}$. By concavity, the function $\Pi(\mathbf{x}) := \sum_{i \in I} \pi_i(x_i)$ is continuous on $\mathbf{A}$. Since $\Pi(\mathbf{x}^k)$ steadily improves,

$$\lim_{k \to \infty} \Pi(\mathbf{x}^k) =: \Pi(\mathcal{L}) = \Pi(\mathbf{x}) \;\text{ for each } \mathbf{x} = (x_i) \in \mathcal{L}.$$

We claim that

$$\lim_{k \to +\infty} \sigma_{ij}^k \mathfrak{S}_{ij}(x_i^k, x_j^k) \left\| d^k \right\| = 0.$$

Otherwise, inserting $d = d^k = P_{ij}(g_i^k - g_j^k)$ in (16) yields the contradiction $\Pi(\mathcal{L}) = +\infty$. Further, *we claim that the sequence* $\mathbf{x}^k$ *is asymptotically regular*, meaning

$$\left\| \mathbf{x}^{k+1} - \mathbf{x}^k \right\| \to 0. \tag{20}$$

To see this, note that Π is Lipschitz continuous on the compact set $\mathbf{A}$. Consequently, its supergradients are norm-bounded there by some $\beta > 0$. Now, if agents i, j are the ones that trade at stage k,

$$\left\| d^k \right\| = \left\| P_{ij}(g_i^k - g_j^k) \right\| \le \left\| g_i^k - g_j^k \right\| \le 2 \max \left\{ \left\| g_i^k \right\|, \left\| g_j^k \right\| \right\} \le 2\beta.$$

Hence assertion (20) follows from $\sigma^k \to 0$ and

$$\left\| \mathbf{x}^{k+1} - \mathbf{x}^k \right\| = \sigma^k 2^{1/2} \left\| d^k \right\| \leq \sigma^k 2^{3/2} \beta.$$

Consider now any limit point $\mathbf{x} = (x_i) \in \mathcal{L}$. Then $\mathbf{x} = \lim_{k \in K} \mathbf{x}^k$ for some subsequence $K \subseteq \{0, 1, ...\}$. If necessary, pick a subsequence of K in which the lapse between consecutive members is greater than l. Assume this is already done.

We claim that $\mathfrak{S}_{ij}(x_i, x_j) = 0$ *for every pair* i, j. By way of contradiction, suppose some agent pair i, j has $\mathfrak{S}_{ij}(x_i, x_j) > 0$. Let $\kappa(k)$ be the first stage $\geq k \in K$ at which i, j trade. Thus emerges, by the quasi-cyclic nature of encounters, a sequence $\mathcal{K}$ of stages $\kappa(k) \in [k, ..., k+l]$, $k \in K$, at which i, j *always* trade. The asymptotic regularity (20) and $\mathbf{x} = \lim_{k \in K} \mathbf{x}^k$ ensure that $\mathbf{x} = \lim_{\kappa \in \mathcal{K}} \mathbf{x}^\kappa$. By Proposition 4.2 and the lower semicontinuity of $\mathfrak{S}_{ij}$, we must have

$$\lim_{\kappa \in \mathcal{K}} \inf \|d^\kappa\| = \lim_{\kappa \in \mathcal{K}} \inf \left\| P_{ij}(g_i^\kappa - g_j^\kappa) \right\| \geq \lim_{\kappa \in \mathcal{K}} \inf \mathfrak{S}_{ij}(x_i^\kappa, x_j^\kappa) \geq \mathfrak{S}_{ij}(x_i, x_j) > 0.$$

Now (19) yields $\sum_{\kappa \in \mathcal{K}} \sigma_{ij}^\kappa = \infty$. Invoking the lower bound in (16), this again implies $\Pi(\mathcal{L}) = +\infty$. We conclude that $\mathfrak{S}_{ij}(x_i, x_j) = 0$ for each pair i, j. Hence $\mathbf{x} \in \mathbf{C}$. $\square$

Theorem 5.2 (On complete trade and equilibrium). *Suppose, at any profile* $\mathbf{x} = (x_i) \in \mathbf{C}$, *that at least one agent* $i = i(\mathbf{x})$ *has* π_i *differentiable at* x_i *and* $x_i \in intX_i$. *Then, each point* $\mathbf{x} \in \mathbf{C}$ *is part of an equilibrium.*

Proof. For any $\mathbf{x} \in \mathbf{C}$ let $i(\mathbf{x})$ be an agent for which $p := \pi_i'(x_i)$ is unique with $x_i \in intX_i$. Then the equilibrium price becomes unique. That is, p is the unique solution to (9). $\square$

6. SOME NUMERICAL ILLUSTRATIONS

This section brings out a few simple examples, each simulating the exchange process on a computer. To illustrate and probe the great flexibility in choice of directions, protocol, and step-sizes, we have opted for much additional freedom. To wit, in the subsequent numerical instances, we circumvent - or simply ignore - some requirements imposed above.

Specifically, for convience, we *never* control that condition (16) holds. Instead, agent i, j simply select *any* shadow prices alias dual solutions or supergradients $g_i \in \partial u_i(x_i)$, $g_j \in \partial u_j(x_j)$. We also relax the quasi-cyclic regime. Instead, when $I = \{1, ..., n\}$, in each complete round agent i meets agents $i \pm 1$ (modulo n) once and only these two. Nonetheless, as seen below, convergence obtains all the same.

In all examples, since x_i is construed as an input vector, we naturally require that $x_i \geq 0$; that is, each $\mathbb{X}_i = \mathbb{X}_+ = \mathbb{R}_+^C$. Consequently, X_i has closed tangent cone

$$D(X_i, x) = D_i x = \Pi_{c \in C} D(\mathbb{R}_+, x_c)$$

so that

$$D_{ij}(x_i, x_j) = \Pi_{c \in C} \left[D(\mathbb{R}_+, x_{ic}) \cap -D(\mathbb{R}_+, x_{jc}) \right].$$

It follows that $d = (d_c) = P_{ij}\Delta$ where d_c is the closest approximation to Δ_c in the closed "interval" $D(\mathbb{R}_+, x_{ic}) \cap -D(\mathbb{R}_+, x_{jc})$.

The chosen instance $\mathbb{X}_i = \mathbb{X}_+ = \mathbb{R}_+^C$ makes it easy to calculate *maximal step-sizes*

$$\bar{\sigma} = \sigma(x_i, x_j, d) := \min \{\sigma_i(x_i, d), \sigma_j(x_j, -d)\} .$$

where

$$\sigma_i(x_i, d) := \min \{-x_{ic}/d_c : d_c < 0\} \quad \text{and} \quad \sigma_j(x_j, -d) := \min \{x_{jc}/d_c : d_c > 0\} .$$

Agents are allowed to take maximal steps for a deliberately long lapse. If such a step does not improve or maintain the joint payoff, parties i, j may invoke the standard bisection method to ensure that $\Delta \pi_{ij} \geq 0$.

Further, for long-term stability - in view of hypotheses (18) and (19) - we select a priori a sequence $\{\hat{\sigma}_k\}$ of positive step-sizes, satisfying

$$\hat{\sigma}_k \to 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \hat{\sigma}_k = +\infty.$$

Thus, in the main, whenever agents i, j trade at some stage k, they employ step-size $\sigma_k := \min \{\hat{\sigma}_k, \ \bar{\sigma}_k\}$.

Coming finally to numerical instanes, there are always three agents: $I = \{1, 2, 3\}$, two inputs: $\#C = 2$, and three productive activities: $\dim \mathbb{Y}_i = 3$.

Example 1 Suppose the agents have equal endowments $e_i = (1, 1)$ and objectives $y_i^* = (1, 1, 1)$, but different technology matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}, \ A_2 = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \tag{21}$$

Step-sizes $\sigma_k = \min\{\bar{\sigma}_k, 10/(1 + 2k)\}$ yield trajectories x_i^k :

$k\backslash i$	1	2	3	$\sum_{i \in I} \pi_i$
0	$(1.00, 1.00)$	$(1.00, 1.00)$	$(1, 1)$	1.17
1	$(0.58, 1.42)$	$(1.43, 0.58)$	$(1, 1)$	1.44
10	$(0.48, 1.52)$	$(1.52, 0.48)$	$(1, 1)$	1.45
50	$(0.50, 1.50)$	$(1.50, 0.50)$	$(1, 1)$	1.49
200	$(0.50, 1.50)$	$(1.50, 0.50)$	$(1, 1)$	1.50

Note that already after 10 barters $\sum_{i \in I} \pi_i$ is within 4% from its optimal value 1.5. The optimal allocation is not unique; agent 3 might just as well have used all resources. $p = (0.5, 0.5)$ is a shadow price. As expected, allocations tend to stabilize more

slowly:
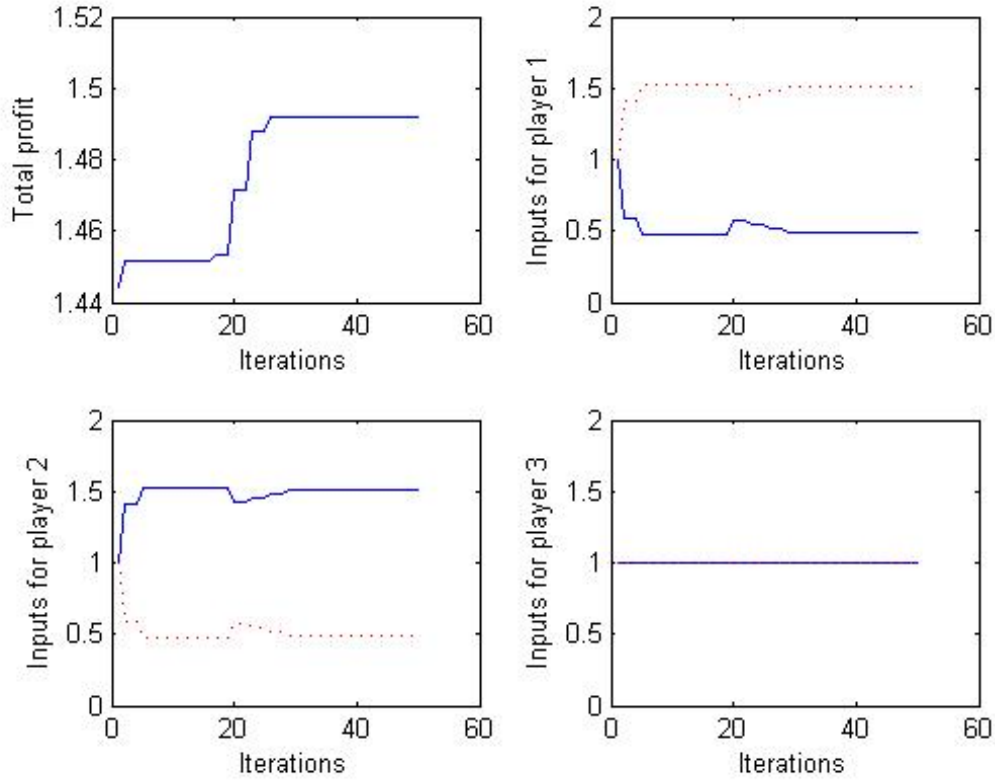
Trajectories for Example 1

Example 2. Clearly, letting some agent use each inputs more efficiently, he will be the only active producer. To see this, keep the data of Example 1 except the matrices, now replaced with

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \ A_3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Using step-sizes $\sigma_k = \min\{\bar{\sigma}_k, 100/(1+k)$, convergence of x_i^k obtains quickly:

$k \backslash i$	1	2	3	π_I
0	$(1,1)$	$(1,1)$	$(1,1)$	1.83
1	$(2,2)$	$(0,0)$	$(1,1)$	2.33
2	$(2,2)$	$(1,1)$	$(0,0)$	2.49
3	$(2,2)$	$(1,1)$	$(0,0)$	2.49
4	$(3,3)$	$(0,0)$	$(0,0)$	2.99
20	$(3,3)$	$(0,0)$	$(0,0)$	3.00

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