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FEASIBILITY IN FINITE TIME

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Abstract

It is common to tolerate that a system's performance be unsustainable during an interim period. To live long however, its state must eventually satisfy various constraints. In this regard we design here differential inclusions that generate, in one generic format, two distinct phases of system dynamics. The first ensures feasibility in finite time; the second maintains that property forever after.

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1 Introduction

To make sense, or simply to survive, a constrained system must, sooner or later, evolve within some, maybe moving subset of the ambient state space. In other words: the dynamics had better become sustainable and the system itself *viable*. Concerns with viability have spurred substantial development in system theory.

Set-valued analysis has thereby acquired a key role. Notably, set-valued differentials turn out useful for control and design of processes geared at feasibility, optimality, or stability.[1] Important examples include subgradient

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[1] For comprehensive studies consult [2], [4], [5], [16], [17], [22].

projection and adaptive play among noncooperative agents. Continuous-time, deterministic versions of such processes often assume the generic form

$$\dot{x}(t) \in M(t, x(t)) - P(t, x(t)). \tag{1}$$

Here $M(t, x(t))$ and $P(t, x(t))$ are subsets of a real Banach space $\mathbb{X}$. Typically, M is the major moving force, and often monotone, while P is a penalty term appended to ensure or eventually maintain feasibility.

By a *solution* to (1) is understood an absolutely continuous profile $t \in [0, T[\mapsto x(t) \in \mathbb{X}$, which starts at a specified initial point $x(0)$, extends up to possibly maximal time $T \in]0, +\infty]$, and satisfies (1) almost everywhere (a.e.). Several studies address existence and uniqueness of solutions; see [2], [4], [5], [7], [17], [22]. Such issues are however, not discussed here. In fact, we shall, in the main, simply presume existence, ignore uniqueness, and rather explore the following problem:

Suppose $x(t)$ must hit a nonempty closed set $S(t) \subset \mathbb{X}$ within a prescribed time limit, and follow $S(t)$ forever after. It may well happen that $x(0) \notin S(0)$. Then, can some proper choice $P(t, x(t))$ forces $x(t)$ permanently into $S(t)$ within critical time?

Put differently: beyond some deadline the state should perfectly track a moving set or state space $S(t)$. For brevity declare $x(t)$ *feasible* if $x(t) \in S(t)$. Our chief purpose is to bring out constructive procedures that eventually make the state feasible. Not surprisingly, the distance

$$d_S(x) := d(x, S) := \inf \{\|x - s\| : s \in S\}$$

from $x = x(t)$ to $S = S(t)$ will be instrumental. That entity is already remarkable in several ways. Besides being central in nonsmooth analysis it facilitates analysis and design of exact penalty methods in optimization. Further, it relates to the geometry of closed sets and defines topologies on such [6]. Seemingly less known however, is its applicability in system dynamics that cannot violate feasibility for long time.

Because the distance and other auxiliary functions are nonsmooth, generalized subdifferentials must here take the place of gradients. However, to make the paper accessible, little knowledge or review of nonsmooth analysis is required. This fits the very simple and basic idea, namely: Estimates of the time to absorption could derive from steadily reducing the distance to feasibility. Such reduction is achievable in various ways. Sections 3&4 use anti-gradients of $d_{S(t)}(x)$; Section 5 invokes flows that aim at feasibility, and Section 6, upon presuming that $S(t) = S = \{f \leq 0\}$ for some lsc function $f : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$, employs steepest-descent like methods.

After preliminary observations, following shortly, Section 3 handles instances with possibly non-regular subsets $S(t)$. Section 4 presumes regular sets. Sections 5 considers convex-like cases, and, as said, Section 6 deals with stationary sublevel sets of the form $\{f \leq 0\}$.

2 Preliminaries

Intuition tells that some part $P(t, x(t))$ of a *normal cone* $N_{S(t)}(x(t))$ to $S(t)$ at $x(t)$ may serve well if already $x(t) \in S(t)$.[2] But outside $S(t)$ another force must be put to work. That force should there reduce the *distance*

$$D(t) := d_{S(t)}(x(t))$$

from $x(t)$ to feasibility. On that account we shall be guided by the following auxiliary result. To simplify its statement, and to avoid repetitions, let henceforth $\delta : [0, T[\to \mathbb{R}_+$ *be Lebesgue integrable with* $\lim_{t \nearrow T} \int_0^t \delta > D(0)$, *and define a deadline* $\bar{t} = \bar{t}(D(0), \delta)$ *implicitly by*

$$\bar{t} := \inf \left\{ t : D(0) \leq \int_0^t \delta \right\}. \tag{2}$$

Lemma 1. (On finite time absorption) *Let* $t \rightrightarrows S(t)$ *be outer continuous on* $[0, T[$. *Suppose* $D(\cdot)$ *is absolutely continuous along a solution* $x(\cdot)$ *of* (1) *with* $\dot{D}(t) \leq -\delta(t)$ *almost whenever* $x(t) \notin S(t)$. *Then, if the state is feasible at some instant, it remains so thereafter. Moreover, the state will become feasible no later than time* $\bar{t}$. *On no proper interval is* $\dot{D} > 0$ *a.e.*

Proof. Suppose $x(t) \notin S(t)$ at time $t > 0$, but the state was already feasible at some prior time $\tau \in [0, t[$. On that assumption let

$$t^- := \sup \{ \tau \in [0, t] : x(\tau) \in S(\tau) \}.$$

Given $S(\cdot)$ outer continuous and $x(\cdot)$ continuous, it follows that $t^- < t$. This yield the absurdity

$$0 < D(t) = D(t^-) + \int_{t^-}^t \dot{D} \leq D(t^-) - \int_{t^-}^t \delta \leq D(t^-) = 0.$$

Similarly, if $x(\tau) \notin S(\tau)$ for all $\tau \in [0, t]$, then

$$0 < D(t) = D(0) + \int_0^t \dot{D} \leq D(0) - \int_0^t \delta,$$

[2] When $M(t, x(t)) = \{0\}$ and $P(t, x(t))$ is the Clarke normal cone to $S(t)$, we get a so-called *sweeping process* studied by [7], [12], [24], [25], [26], [30] and others.

this implying $\int_0^t \delta < D(0)$, whence $t < \bar{t}$. If $\dot{D} > 0$ on some proper interval $[t_-, t_+] \subset [0, T[$, we can take $x(t_-)$ feasible to get $D(t_+) > 0$ and thereby contradict the feasibility of $x(t_+)$. $\square$

In short, what imports is to have $D(\cdot)$ absolutely continuous and $\dot{D}(\cdot)$ almost always sufficiently negative while $x(t) \notin S(t)$. For the sake of absolute continuity henceforth suppose $S(t)$ moves so smoothly that

$$\left| d_{S(\tau)}(x) - d_{S(t)}(x) \right| \leq |\vartheta(\tau, x) - \vartheta(t, x)| \tag{3}$$

when $\tau, t \geq 0$, and $x \in \mathbb{X}$. Here, by assumption, $\vartheta : \mathbb{R}_+ \times \mathbb{X} \to \mathbb{R}$ satisfies

$$\lim_{\tau \to t} \frac{\vartheta(\tau, x(\tau)) - \vartheta(t, x(\tau))}{\tau - t} = \frac{\partial}{\partial t} \vartheta(t, x) \tag{4}$$

whenever $t \geq 0$, and an absolute continuous $x(\tau)$ converges to x as $0 \leq \tau \to t$.

Further, to have $\dot{D}(t)$ almost always sufficiently negative while $x(t) \notin S(t)$, just here, for simplicity in argument, assume $\mathbb{X}$ finite-dimensional. Then the *antigradient* $-\nabla d_{S(t)}(\cdot)$, if any, would serve well because it points in direction of steepest distance descent. The trouble is that $\nabla d_{S(t)}(\cdot)$ often fails to exist. Gradient methods have great appeal though. They tempt us to replace ∇ by a subdifferential ∂ and posit

$$P(t, x) := \mu(t, x) \partial d_{S(t)}(x), \tag{5}$$

where $\mu(t, x) \geq 0$ is a speed or scale factor, and ∂ a suitable subdifferential. In terms of geometry and *projection*

$$\Pi_{S(t)}(x) := \left\{ s \in S(t) : \|x - s\| = d_{S(t)}(x) \right\},$$

with $\mathbb{X}$ is finite-dimensional and ∂ the *Fréchet* or *Mordukhovich subdifferential*,

$$\partial d_{S(t)}(x) = \frac{x - \Pi_{S(t)}(x)}{d_{S(t)}(x)} \quad \text{for} \quad x \notin S(t); \tag{6}$$

see Proposition 2.1 in [21] and Example 8.53 in [28].

Equation (6) tells *three* things. First, each anti-subgradient, belonging to $-\partial d_{S(t)}(x)$ at $x \notin S(t)$, has unit length, and it points from x towards a best approximation in $S(t)$. Second, regular subdifferentiability of $d_{S(t)}(\cdot)$ obtains at $x \notin S(t)$ when $\Pi_{S(t)}(x)$ reduces to a singleton. Third, outside $S(t)$ the *Clarke subdifferential*

$$\partial^C d_{S(t)}(x) = \frac{conv \left\{ x - \Pi_{S(t)}(x) \right\}}{d_{S(t)}(x)}$$

may often appear somewhat "vague" by including vectors that do not point "perpendicularly" towards $S(t)$.

In finite dimensions these observations lead us to employ (5) as a major vehicle - and to favor application of the Fréchet or Mordukhovich ∂. Further, they speak for $\mu(t,x)$ being large enough in (5) to majorize

$$\|M(t,x)\| := \sup\{\|m\| : m \in M(t,x)\}$$

plus $\left|\frac{\partial}{\partial t}\vartheta(t,x)\right|$ if $S(t)$ moves, and still provide reduced distance. In sum, along a solution $x(\cdot)$ of (1), while $x = x(t) \notin S(t)$, we shall require that

$$\mu(t,x)d^2(0, \partial d_{S(t)}(x)) \geq \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| + \delta(t) + \|M(t,x)\| \quad \text{a.e.} \tag{7}$$

Broadly (7) requires that $\mu(t,x)$ offsets drift of $S(t)$, accounts for the desired rate of distance decay $\delta(t)$, and, in addition, dominates all outward oriented forces in $M(t,x)$ if any. When $\delta(t) > 0$, inequality (7) couldn't possibly hold if $0 \in \partial d_{S(t)}(x)$ for some $x \notin S(t)$. It is paramount therefore, that $d_{S(t)}(x)$ has a non-zero "slope" at each non-feasible x; see Lemmata 2&3 and (11).

One may let $\delta(t)$ depend on the state $x(t)$ as well. The important requirement remains though, that $\int_0^{T-} \delta(t, x(t))dt > D(0)$. Clearly, a specification of that last sort makes it harder to estimate the absorption time $\bar{t}$.

There are good reasons to push beyond finite-dimensional settings. Then any ∂ maps into the dual space $\mathbb{X}^*$. Accordingly, to make $\partial d_{S(t)}(\cdot)$ part of a primal force, it must be brought back via a *duality mapping*

$$\mathcal{D}x^* := \left\{x \in \mathbb{X} : \langle x^*, x\rangle = \|x^*\|^{*2} = \|x\|^2\right\}$$

from $\mathbb{X}^*$ into its predual $\mathbb{X}$. This done, $\mathcal{D}\partial d_{S(t)}$ is apt to work well. So, apart from Sections 5 & 6, we let (1) assume the form

$$\dot{x} \in M(t,x) - \mu(t,x)\mathcal{D}\partial d_{S(t)}(x). \tag{8}$$

With $\mathbb{X}$ reflexive and $\|\cdot\|$ strictly convex, $\mathcal{D}$ becomes most amiable, being then single-valued and globally defined. And clearly, if $\mathbb{X}$ is Hilbert, one can dispense with $\mathcal{D}$. We find the greater generality of Banach spaces worthwhile though, to see precisely where some key arguments must be qualified. As one might expect, differentiability properties of the norm will become crucial.

The penalty term $P(t,x) = \mu(t,x)\mathcal{D}\partial d_{S(t)}(x)$ separates, in multiplicative manner, the direction $\in \mathcal{D}\partial d_{S(t)}(x)$ from the speed $\mu(t,x)$ along that line. There is of course considerable latitude in choosing the latter, but taken together these offer a closed loop, feed-back control that aims at permanent feasibility.

3 Tracking Non-Regular Sets

For system (8) to aim at feasibility, the selection from $\mathcal{D}\partial d_{S(t)}(x)$ would do well by being "perpendicular" or almost normal to $S(t)$ at some best feasible approximation. With $\mathbb{X}$ finite-dimensional and a Fréchet or Mordukhovich ∂, formula (6) already displays that property. To pursue and extend this geometric perspective, we fix S for a while and refer, for any $\varepsilon \geq 0$, to

$$N_\varepsilon^F(x, S) := \left\{ x^* \in \mathbb{X}^* : \limsup_{x' \in S \to x} \frac{\langle x^*, x' - x \rangle}{\|x' - x\|} \leq \varepsilon \right\}$$

as the set of (*Fréchet*) *ε-normals* to S at $x \in S$. When the parameter ε is nil, it requires no mention, and elements of $N^F(x, S) := N_0^F(x, S)$ are simply called *Fréchet normals*. The *Mordukhovich* (basic or limiting) *normal cone* N [22] emerges via a *weak** sequential outer limit as

$$N(x, S) := \left\{ x^* \in \mathbb{X}^* : \exists x_k \in S \to x,\ x_k^* \in N_{\varepsilon_k}^F(x, S) \to^{w^*} x^*,\ \text{and } \varepsilon_k \to 0^+ \right\}.$$

Derived from that cone N is a *Mordukhovich subdifferential* $\partial^{\mathcal{M}}$ that operates on functions $f : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$ and points $x \in dom f := f^{-1}(\mathbb{R})$. As usual, it is defined in terms of the epigraph $epi f := \{(x, r) \in \mathbb{X} \times \mathbb{R} : f(x) \leq r\}$ by

$$\partial^{\mathcal{M}} f(x) := \{x^* \in \mathbb{X}^* : (x^*, -1) \in N((x, f(x)), epi f)\}.$$

Much simpler is the definition of the *Dini subdifferential*:

$$\partial^D f(x) := \left\{ x^* \in \mathbb{X}^* : \liminf_{h \to 0^+} \frac{f(x + hv) - f(x)}{h} \geq \langle x^*, v \rangle \ \ \forall v \right\}.$$

And so is that of the *Fréchet subdifferential*:

$$\begin{aligned} \partial^F f(x) &:= \left\{ x^* \in \mathbb{X}^* : \liminf_{x' \to x} \tfrac{f(x') - f(x) - \langle x^*, x' - x \rangle}{\|x' - x\|} \geq 0 \right\} \\ &= \left\{ x^* \in \mathbb{X}^* : (x^*, -1) \in N^F((x, f(x)), epi f) \right\}. \end{aligned}$$

f is declared *Mordukhovich regular* at $x \in dom f$ if $\partial^{\mathcal{M}} f(x) = \partial^F f(x)$ and *Clarke regular* there if $\partial^C f(x) = \partial^D f(x)$.

Of chief interest here is regularity of the distance function. In finite dimensions, Clarke regularity of that particular function coincides with Mordukhovich regularity, and it amounts to the requirement that the Euclidean projection on the set at hand be a singleton. With infinite dimensions, Clarke regularity of the distance function does not imply its Fréchet-normal regularity. In the reflexive case, the Fréchet normality is equivalent to the Mordukhovich one of the distance function; see [10] and references therein.

Important characterizations for the Hilbert setting are given in [27].

To abbreviate some repeated statements, for any subdifferential ∂, *we write* $\partial \subseteq \partial^C$ *to signal that* $\partial l(x) \subseteq \partial^C l(x)$ *whenever* $l : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$ *is finite-valued and locally Lipschitz near* x.[3] Further, it is tacitly assumed that ∂ coincides with the customary subdifferential of convex analysis when operating on a convex function $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$.

Theorem 1. (Finite time absorption) *Choose any subdifferential* $\partial \subseteq \partial^C$. *Suppose* $x(\cdot)$ *solves* (8) *on* $[0, T[$. *Also suppose* $-d_{S(t)}(\cdot)$ *is Clarke regular outside* $S(t)$. *Then, under condition* (7), $x(t) \in S(t)$ *for each* $t \geq \bar{t}$.

The proof derives directly from Lemma 1 after invoking two auxiliary results, the following being of independent interest.

Lemma 2. (Derivative of the distance in anti-subgradient direction) *Fix here a nonempty closed stationary* $S \subset \mathbb{X}$, *and choose any* $\partial \subseteq \partial^C$. *Suppose* $v \in \mathcal{D}\partial d_S(x)$, *and that* $-d_S(\cdot)$ *is Clarke regular at* x. *Then* $\|v\| \leq 1$, *and*

$$-\|v\| \leq \lim_{h \to 0^+} \frac{d_S(x - hv) - d_S(x)}{h} \leq -\|v\|^2 . \tag{9}$$

If moreover, $v \neq 0$, *then* $x \notin S$.

Proof. Let $d_S^C(x; v)$ denote the Clarke directional derivative of $d_S(\cdot)$ at x in direction v. Choose a subgradient $g \in \partial d_S(x)$ such that $v \in \mathcal{D}g$. Because $\partial d_S(x) \subseteq \partial^C d_S(x)$ we get $\|v\|^2 = \langle g, v \rangle \leq$

$$d_S^C(x; v) = \sup_{v^* \in \partial^C d_S(x)} \langle v^*, v \rangle = \sup_{v^* \in \partial^C(-d_S(x))} \langle v^*, -v \rangle = \sup_{v^* \in \partial^D(-d_S(x))} \langle v^*, -v \rangle$$

$$\leq \liminf_{h \to 0^+} \frac{-d_S(x - hv) + d_S(x)}{h} = -\limsup_{h \to 0^+} \frac{d_S(x - hv) - d_S(x)}{h}.$$

Consequently,

$$\limsup_{h \to 0^+} \frac{d_S(x - hv) - d_S(x)}{h} \leq -\|v\|^2 .$$

Also, because $d_S(x) \leq d_S(x - hv) + h\|v\|$,

$$\limsup_{h \to 0^+} \frac{-d_S(x - hv) + d_S(x)}{h} \leq \|v\| .$$

[3] What we need is merely that $\partial d_S(x) \subseteq \partial^C d_S(x)$ whenever $S \subseteq \mathbb{X}$ is closed and $x \in \mathbb{X}$.

Invoking once again the Clarke regularity of $-d_S(\cdot)$ at x, all the $\liminf$ and $\limsup$ in this proof are customary limits. Thus (9) follows and thereby $\|v\| \leq 1$. Moreover, because $d_S \geq 0$, if $v \neq 0$, the rightmost inequality in (9) implies $x \notin S$. $\Box$

With $\mathbb{X}$ finite-dimensional and $\partial \in \{\partial^F, \partial^{\mathcal{M}}\}$, equation (6) already told that $\|v\| = 1$ when $v \in \mathcal{D}\partial d_S(x)$ and $x \notin S$. When $\partial = \partial^F$ the same property holds in any Banach space; see Theorem 1.99 in [22]. It is also valid with $\mathbb{X}$ Asplund and $\partial = \partial^D$.

Still with $\mathbb{X}$ Asplund, $\partial^C \subseteq clconv\partial^{\mathcal{M}}$, and it would suffice in Lemma 2 that $-d_S(\cdot)$ be Mordukhovich regular at x.

Lemma 3. (Monotone approach to feasibility) *Choose any subdifferential $\partial \subseteq \partial^C$. Consider a solution $x(\cdot)$ to* (8) *at a time $t \geq 0$ and non-feasible state $x = x(t)$ where $-d_{S(t)}(\cdot)$ is Clarke regular, all time derivatives $\dot{D}(t)$, $\frac{\partial}{\partial t}\vartheta(t,x)$, $\dot{x} = \dot{x}(t)$ exist, and* (7) *applies. Then $\dot{D}(t) \leq -\delta(t)$.*

Proof. For $\tau > t$ and $h := \tau - t$ the differentiability of $x(\cdot)$ at $x = x(t)$ yields

$$x(\tau) = x + \dot{x}h + \varepsilon(h)B$$

where B denotes the closed unit ball in $\mathbb{X}$, and $\varepsilon(h)/h \to 0$ as $h \to 0$. Further, because $d_{S(t)}(\cdot)$ is Lipschitz with modulus 1,

$$\frac{D(\tau) - D(t)}{\tau - t} = \left\{d_{S(\tau)}(x(\tau)) - d_{S(t)}(x(\tau)) + d_{S(t)}(x(\tau)) - d_{S(t)}(x(t))\right\}/h$$

$$\leq \frac{1}{\tau - t}\left|\vartheta(\tau, x(\tau)) - v(t, x(\tau))\right| + \frac{1}{h}[d_{S(t)}(x + h\dot{x}) - d_{S(t)}(x) + \varepsilon(h)].$$

By the said Lipschitz continuity

$$d_{S(t)}(x + h\dot{x}) \leq d_{S(t)}(x + h(\dot{x} - m)) + h\left\|m\right\|.$$

Here $\dot{x} - m = -\mu v$ with $m \in M(t,x)$, $v \in \mathcal{D}\partial d_S(x)$ and $\mu \geq 0$. Consequently, Lemma 2 gives

$$\lim_{h \to 0^+} \frac{d_{S(t)}(x - h\mu v) - d_{S(t)}(x)}{h} \leq -\mu\left\|v\right\|^2 \leq -\mu d^2(0, \partial d_{S(t)}(x))$$

hence, because $x \notin S(t)$ and (7) applies,

$$\dot{D}(t) \leq \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| - \mu d^2(0, \partial d_{S(t)}(x)) + \|m\| \leq -\delta(t). \quad \Box$$

A Banach space is declared *uniformly Gâteaux smooth* if its norm is uniformly Gâteaux differentiable on the unit sphere.

Theorem 2. *Suppose $\mathbb{X}$ is uniformly Gâteaux smooth. Choose any subdifferential $\partial \subseteq \partial^C$. Suppose $x(\cdot)$ solves* (8) *on* $[0, T[$. *Then, under condition* (7), *$x(t) \in S(t)$ for each $t \geq \bar{t}$.*

Proof. In this case $-d_S(\cdot)$ has a Gâteaux derivative that coincides with $(-d_S)^C(x;)$ whenever $x \notin S$; see [9]. □

Note that (7) came into play while the state still stayed infeasible. By contrast, for the purpose of maintained feasibility it suffices that

$$\mu(t,x) \geq \left| \frac{\partial}{\partial t} \vartheta(t,x) \right| + \|M(t,x)\| \text{ almost whenever } x \in S(t). \tag{10}$$

Under this proviso, once $x(t)$ becomes feasible, it will remain so with $\delta = 0$ from that moment onwards. So, the dynamics have two phases: a first and transient period, if any, is followed by a subsequent viable regime [5]. In many settings, $\partial d_S(x)$ is part of the unit sphere when $x \notin S$; see e.g. (6) or Theorem 1.99 in [22]. On such occasions (7) amounts of course to

$$\mu(t,x) \geq \left| \frac{\partial}{\partial t} \vartheta(t,x) \right| + \delta(t) + \|M(t,x)\| \quad \text{almost whenever } \ x \notin S(t).$$

In any case, a larger δ brings about earlier absorption. Further, if $x(0) \notin S(0)$, equation (2) already tells that δ must be positive during some transient time lapse. To see the same thing differently, let for example, S be stationary convex in a Hilbert space, $M = \partial d_S$, and

$$\mu(t,x) d^2(0, \partial d_{S(t)}(x)) = \mu(t,x) = \delta(t) + \|M(x)\| = \delta(t) + 1$$

for $x \notin S$. Then, $\dot{x}(t) = 0$ if $\delta(t) = 0$ while $x(t) \notin S$, to the effect that absorption never happens.

It makes, of course, a difference which ∂ operates in (8). For example, let $M = \{0\}$, fix $S = \{x \in \mathbb{R}^2 : x_2 \geq -|x_1|\}$, and posit $\mu = 1$, to have the inclusion $\dot{x} \in -\partial d_S(x)$. With $x(0) = (-\sqrt{2}, 0)$ the subdifferential $\partial^{\mathcal{M}}$ brings the state to hit bdS where $|x_1| = 1$ at time 1. Alternatively, if ∂^C is at the steering wheel, the state may encounter bdS any place where $|x_1| \leq 1$ at a time $t \in [1, 1 + \sqrt{2}]$. This S isn't regular at the origin.

Fairly often, and quite naturally, $\partial^{\mathcal{M}}$ is small but non-convex. As a result, (8) generates several trajectories. To illustrate, fix S as a finite union of disjoint closed sets in a Hilbert space. Posit $M = \{0\}$, $\mu = 1$, and $x(0) \notin S$. Then, with $\partial = \partial^{\mathcal{M}}$ (8) reads $\dot{x} \in -\partial^{\mathcal{M}} d_S(x)$, an inclusion that generates precisely as many trajectories (each rectilinear) as $x(0)$ has closest feasible points.

4 Pursuing Regular Sets

This section specializes in two ways. For one, the space $\mathbb{X}$ *is now real Hilbert with inner product* $\langle \cdot, \cdot \rangle$. For the other, sets $S(t)$ *are here presumed regular.*

Operating in such a setting brings several advantages: *First*, the duality mapping $\mathcal{D}$ can be dispensed with. *Second*, any $x(\cdot)$ which is absolutely continuous on an interval $[\tau, t]$ satisfies $x(t) = x(\tau) + \int_\tau^t \dot{x}$. *Third*, proofs becomes simpler and more direct.

When $\partial^F f(x)$ coincides with the *Clarke subdifferential* $\partial^C f(x)$, we say that f is *subdifferentially regular* at x. Such regularity of the extended indicator I_S amounts to have the Fréchet normal cone $N_S^F(x)$ coincide with the *Clarke normal cone* $N_S^C(x)$ at $x \in S$. When so happens, we declare S *normally regular* at its member x. Bounkhel and Thibault [10] show that this property is equivalent to the subdifferential regularity of the distance function $d_S(\cdot)$ at $x \in S$. When normal regularity prevails at each of its points, we simply say that S is normally regular.

Recently, Clarke, Stern and Wolenski [15] characterized closed subsets S of $\mathbb{X}$ for which $d_S(\cdot)$ is continuously differentiable on $S + \beta B$ for some positive β, B being the closed unit ball. Local versions of such differentiability outside S have later been studied by Poliquin, Rockafellar and Thibault [27]. Their results are useful for proving the next:

Theorem 3. (Finite time absorption into a moving, maybe non-convex set) *Suppose* $d_{S(t)}(\cdot)$ *is subdifferentially regular along a solution* $x(\cdot)$ *to* (8) *with* $\partial \subseteq \partial^C$. *Then* (7) *entails that* $x(t) \in S(t)$ *for all* $t \geq \bar{t}$.

Proof. Assumption (3) ensures that $D(\cdot)$ is absolutely continuous. Fix any time $t \geq 0$ at which $x := x(t) \notin S(t)$. Suppose $\dot{D}(t)$, $\frac{\partial}{\partial t}\vartheta(t, x)$ and $\dot{x}(t)$ all exist. Then, for $\tau > t$ we have

$$\begin{aligned} D(\tau) - D(t) &= d_{S(\tau)}(x(\tau)) - d_{S(t)}(x) \\ &= d_{S(\tau)}(x(\tau)) - d_{S(t)}(x(\tau)) + d_{S(t)}(x(\tau)) - d_{S(t)}(x) \\ &\leq |\vartheta(\tau, x(\tau)) - \vartheta(t, x(\tau))| + d_{S(t)}(x(\tau)) - d_{S(t)}(x). \end{aligned}$$

From Poliquin, Rockafellar, and Thibault [27] we know that $d_{S(t)}(\cdot)$ has a Fréchet derivative x^* at x with $\|x^*\| = 1$. Divide the preceding inequality by

$\tau - t$ and let $\tau \to t^+$ to obtain

$$\begin{aligned}\dot{D}(t) &\le \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| + \langle x^*, \dot{x}(t)\rangle \\ &\le \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| + \|M(t,x)\| - \mu(t,x) \le -\delta(t).\end{aligned}$$

The conclusion now follows from Lemma 1. $\square$

We remark that Theorem 3 holds beyond the Hilbert setting. Indeed, it follows from Proposition 1.5 in [20] that for any normed vector space $\mathbb{X}$ and closed set $S \subset \mathbb{X}$ and $x \notin X$ we have

$$\partial^F d_S(x) \subset \{x^* \in \mathbb{X}^* : \quad \|x^*\| = 1\}. \tag{11}$$

So, Theorem 3 remains valid in uniformly Gâteaux smooth spaces upon replacing condition (7) by

$$\mu(t,x) \ge \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| + \delta(t) + \|M(t,x)\| \quad \text{a.e.}$$

Once feasibility obtains, to preserve it, condition (10) suffices. To the same end, one might posit $P = N^C$ in (1), but doing so offers no advantages:

Proposition 1. (Coincidence of trajectories and viability[4]) *In* (1) *let* $P(t,\cdot) = N^C_{S(t)}(\cdot)$ *be the Clarke normal cone to* $S(t)$ *and suppose* $x(\cdot)$ *solves* (8) *with* $x(0) \in S(0)$. *Also suppose that* $d_{S(t)}(\cdot)$ *is subdifferentially regular along the solution trajectory, and* $\dot{x} = m - n$ *with*

$$m \in M(t,x), \ \ n \in N^C_{S(t)}(x) \ \ \textit{and} \ \ \langle m, n\rangle \le 0 \ \ \textit{a.e.}$$

Then the same $x(\cdot)$ *also solves*

$$\dot{x} \in M(t,x) - \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| \partial^F d_{S(t)}(x).$$

Proof. We shall argue as Thibault [30]. Fix any time $t > 0$ at which $x = x(t)$ and $\vartheta(\cdot, x)$ are differentiable with $n = n(t) \neq 0$. The regularity of $S(t)$ at x tells that

$$\frac{n}{\|n\|} \in N^F_{S(t)}(x) \cap \{x^* : \|x^*\| = 1\} \subset N^F_{S(t)}(x) \cap \{x^* : \|x^*\| \le 1\} = \partial^F d_{S(t)}(x).$$

[4] For more on equivalent viable systems see [11].

Consequently, for any positive ε and time $\tau < t$ sufficiently close to t we have

$$\begin{aligned}\left\langle \frac{n}{\|n\|}, x(\tau) - x(t) \right\rangle &\le d_{S(t)}(x(\tau)) - d_{S(t)}(x(t)) + \varepsilon \|x(\tau) - x(t)\| \\ &= d_{S(t)}(x(\tau)) - d_{S(\tau)}(x(\tau)) + \varepsilon \|x(\tau) - x(t)\| \\ &\le |\vartheta(t, x(\tau)) - \vartheta(\tau, x(\tau))| + \varepsilon \|x(\tau) - x(t)\| .\end{aligned}$$

Thus

$$\left\langle -\frac{n}{\|n\|}, \frac{x(\tau) - x(t)}{\tau - t} \right\rangle \le \left| \frac{\vartheta(t, x(\tau)) - \vartheta(\tau, x(\tau))}{t - \tau} \right| + \varepsilon \left\| \frac{x(\tau) - x(t)}{\tau - t} \right\| .$$

Letting $\tau \nearrow t$ we obtain

$$\left\langle -\frac{n}{\|n\|}, \dot{x}(t) \right\rangle \le \left| \frac{\partial \vartheta(t, x)}{\partial t} \right| + \varepsilon \|\dot{x}(t)\| .$$

Since $\varepsilon > 0$ was arbitrary, and because $\langle m, n \rangle \le 0$, we get from $\dot{x}(t) = m - n$ that

$$\|n\| \le \left| \frac{\partial \vartheta(t, x(t))}{\partial t} \right| ,$$

whence $n \in \left| \frac{\partial \vartheta(t,x)}{\partial t} \right| \partial^F d_{S(t)}(x)$, and the conclusion follows. □

When sets are convex, simpler arguments apply, and some special features merit mention. To divorce separate arguments, first suppose S stationary convex. While $x \notin S$, it holds that $\partial d_S(x) = \{x - \bar{x}\} / \|x - \bar{x}\|$ with $\bar{x} := \Pi_S(x)$. So, omitting repeated mention of time,

$$\begin{aligned} D\dot{D} = \frac{d}{dt}\left[D^2/2\right] &= \langle x - \bar{x}, \dot{x} \rangle \in \left\langle x - \bar{x}, M(x) - \mu(x) \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle \\ &= \langle x - \bar{x}, M(x) \rangle - \mu(x) \|x - \bar{x}\| \\ &\le \{\|M(x)\| - \mu(x)\} \|x - \bar{x}\| \le -\delta \|x - \bar{x}\| = -\delta D. \end{aligned}$$

Consequently, $\dot{D} \le -\delta$ as long as $x \notin S$.

Returning to the setting where $S(\cdot)$ moves, but still is convex, fix $t \ge 0$, posit $x = x(t)$, and let $\tau > t$. By the above argument

$$\lim_{\tau \to t^+} \frac{d_{S(t)}(x(\tau)) - d_{S(t)}(x)}{\tau - t} \le \|M(t, x)\| - \mu(t, x),$$

and one may conclude as in the proof of Theorem 3.

A result of Moreau [23] says that every vector $m \in M(t, x)$ admits a unique, orthogonal decomposition $m = m_{\tan} + m_{nor}$ with $m_{\tan}$ belonging to the standard *tangent cone* of $S(t)$ at $\Pi_{S(t)}(x)$. Closer inspection of the preceding argument reveals that outside $S(t)$ one may contend with the smaller modulus $\mu(t,x(t)) \geq \delta(t) + \|M_{nor}(t,x(t))\|$.

If $M \equiv \{0\}$, S is stationary, and $x(0) \notin S$, then, prior to absorption, system (8) proceeds along the fixed anti-normal vector $\Pi_S(x(0)) - x(0)$. Specifically, provided μ be integrable,

$$x(t) = x(0) + \frac{\Pi_S(x(0)) - x(0)}{D(0)} \int_0^t \mu(\tau, x(\tau))d\tau, \tag{12}$$

and $x(t)$ will eventually hit S at the orthogonal projection $\Pi_S(x(0))$ of the initial point.

Elaborating on the last instance, still with $M = \{0\}$ and S stationary, the particular choice $\mu = \delta$ fits (7) to the effect that $x(t) \notin S$ while $t < \bar{t}$. Thus the time estimate $\bar{t}$ cannot generally be improved.

We conclude this section with a brief mention of prox-regular instances. Recall that x^* is called a *proximal subgradient* to $f : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$ at $x \in dom f$, and we write $x^* \in \partial^P f(x)$, iff there exist positive numbers ρ and σ such that

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle - \sigma \|x' - x\|^2 \text{ whenever } \|x' - x\| \leq \rho.$$

For the particular instance $f = d_S$ it holds at any $x \notin S$ where $\partial^P d_S(x) \neq \emptyset$, that the Fréchet derivative $d'_S(x)$ exists. Moreover, the projection $\Pi_S(x)$ is then a singleton, and

$$\partial^P d_S(x) = d'_S(x) = \frac{x - \Pi_S(x)}{\|x - \Pi_S(x)\|};$$

see Theorem 6.1 in Clarke et al. (1998). Consequently, upon employing $\partial = \partial^P$ in (8), Theorems 1-3 still hold.

5 Convex-like Cases

The query remains that $d_S(\cdot)$ often fails to be regular. In this section let the space $\mathbb{X}$ be finite-dimensional Euclidean. We first explore whether a more abstract system

$$\dot{x} \in V(t,x), \tag{13}$$

with $V(t,x)$ nonempty closed convex, leads towards a stationary S. We say that a set-valued vector field $x \rightrightarrows V(t,x) \subset \mathbb{X}$ *aims towards* $S \subset \mathbb{X}$ *with velocity* $\geq \delta$ *at* $x \notin S$ *if*

$$\sup_{v \in V(t,x)} \inf_{s \in \Pi_S(x)} \langle v, x - s \rangle \leq -\delta. \tag{14}$$

When S is closed, (14) holds iff there exists $\bar{s} \in conv\Pi_S(x)$ such that

$$\sup_{v \in V(t,x)} \langle v, x - \bar{s} \rangle \leq -\delta.$$

To see this, note that the projection $\Pi_S(x)$ is nonempty compact whence so is its convex hull $conv\Pi_S(x)$. Thus, for any given $v \in V(t,x)$ we have

$$\min_{s \in \Pi_S(x)} \langle v, x - s \rangle = \min_{s \in conv\Pi_S(x)} \langle v, x - s \rangle .$$

Let $\bar{s}$ be any point in $conv\Pi_S(x)$ which minimizes the lower semicontinuous function $s \mapsto \sup_{v \in V(t,x)} \langle v, x - s \rangle$ on that set. Since $V(t,x)$ is closed convex, we get by the lop-sided minimax theorem [3] that

$$\begin{aligned} \sup_{v \in V(t,x)} \langle v, x - \bar{s} \rangle &= \min_{s \in conv\Pi_S(x)} \sup_{v \in V(t,x)} \langle v, x - s \rangle \\ &= \sup_{v \in V(t,x)} \min_{s \in conv\Pi_S(x)} \langle v, x - s \rangle = \sup_{v \in V(t,x)} \inf_{s \in \Pi_S(x)} \langle v, x - s \rangle \\ &\leq -\delta. \end{aligned}$$

Recall that an *Euler arc* $0 \leq t \mapsto x(t)$ is the uniform limit of a polygonal curves (i.e. piecewise linear curves), the maximal "mesh sizes" of which tend to zero. Following the arguments in Clarke et al.(1998), Chap.4.2 one may prove the following:

Proposition 2. (Finite-time absorption using an aiming field) *Suppose a closed convex-valued vector field $V(t,x) \neq \varnothing$, with at most linear growth, aims at a fixed closed $S \subset \mathbb{X}$ with velocity $\geq \delta(t,x) d_S(x)$. Then, if $\delta(\cdot,\cdot)$ is continuous, any Euler arc which solves* (13) *on* $[t, \tilde{t}]$ *must satisfy*

$$d_S^2(x(\tilde{t})) - d_S^2(x(t)) \leq -2 \int_t^{\tilde{t}} \delta(\tau, x(\tau)) d_S(x(\tau)) d\tau$$

whence $\dot{D}(t) \leq -\delta(t, x(t))$ a.e. Consequently, $x(t) \in S$ for all $t \geq \bar{t}$. $\square$

The preceding arguments underscore the convenience of dealing with sets which are convex-like somehow. Reflecting on that feature, this section concludes by considering a temporarily fixed, nonempty closed set S that is "not too far from convex". The *Asplund function* $\varphi_S := \left\{ \|\cdot\|^2 - d_S^2 \right\} /2$ now becomes a good instrument.[5] Because $\partial^C(d_S^2/2) = I - \partial^C \varphi_S$ and

[5] It is already prominent in the study of Chebyshev sets; see [1], [18] and references therein.

$\partial^C \varphi_S(x) = conv\Pi_S(x)$, we get $\partial^C(d_S^2/2) = conv\{x - \Pi_S(x)\}$. So, for any $x \notin S$, it follows that $\partial^C d_S(x) =$

$$\partial^C \sqrt{d_S^2(x)} = \frac{\partial^C d_S^2(x)/2}{d_S(x)} = conv\{x - \Pi_S(x)\}/d_S(x) = conv\left\{\frac{x - \Pi_S(x)}{\|x - \Pi_S(x)\|}\right\}.$$

In particular, when $x \notin S$, the subdifferential $\partial^C d_S(x)$ is contained in the unit ball. Define a *convexity modulus* $\kappa_S : S^c \to \mathbb{R}$ of S by

$$\kappa_S(x) := \inf\{\langle x - \bar{x}, x - \hat{x}\rangle : \bar{x}, \hat{x} \in \Pi_S(x)\}/d_S^2(x) \text{ when } x \notin S.$$

Plainly, κ_S assumes values in $[-1, 1]$. And each convex set S has $\kappa_S(\cdot) \equiv 1$ all across the complement S^c of S. Thus, having κ_S close to 1 loosely indicates that S isn't very far from convex. For simplicity let's declare S *convex-like* if $\kappa_S(x) > 0$ whenever $x \notin S$. Anyway, while $x = x(t) \notin S$, dynamics (8), with $\partial \subseteq \partial^C$, yields

$$\begin{aligned} \dot{d}_S(x) &\in \left\langle conv\left\{\frac{x - \Pi_S(x)}{d_S(x)}\right\}, M(t,x) - \mu(t,x) conv\left\{\frac{x - \Pi_S(x)}{d_S(x)}\right\}\right\rangle \\ &\leq \|M(t,x)\| - \mu(t,x)\inf\{\langle x - \bar{x}, x - \hat{x}\rangle : \bar{x}, \hat{x} \in \Pi_S x\}/d_S^2(x) \\ &= \|M(t,x)\| - \mu(t,x)\kappa_S(x) \quad \text{a.e.} \end{aligned}$$

Collecting these observations, and letting $S(t)$ move, we straightforwardly get

Theorem 4. (Reaching convex-like sets in finite time) *Let $\partial \subseteq \partial^C$. Suppose $S(t)$ is convex-like with modulus $\kappa_{S(t)}(\cdot)$. Provided*

$$\kappa_{S(t)}(x) \cdot \mu(t,x) \geq \left|\frac{\partial}{\partial t}\vartheta(t,x)\right| + \delta(t) + \|M(t,x)\|$$

almost whenever $x = x(t) \notin S(t)$, then system (8) *reaches $S(t)$ no later than time $\bar{t}$ and stays in that set forever after.* □

6 Reaching a Sublevel Set

Fix here a nonempty, stationary sublevel set $S := \{f \leq 0\}$, featuring a lower semicontinuous (lsc) function f that maps an Euclidean space $\mathbb{X}$ into $\mathbb{R} \cup \{+\infty\}$. Clearly, f isn't unique. For any continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(r) > 0 \Leftrightarrow r > 0$, the representation $S = \{\varphi \circ f \leq 0\}$ would fit as well. Also, $S = \{d_S \leq 0\}$. We assume though that f is easier to differentiate than d_S.

As before, we seek to steer the state from a known, initial position $x(0)$ towards S. Or, if $x(0) \in S$ already, then one should keep $x(t) \in S$ for all $t > 0$. But now, instead of using the cumbersome distance d_S, we rather want to work with f itself.

To such ends we invoke an *abstract subdifferential* ∂ that associates to any lsc function $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ and point $x \in \mathbb{X}$ a closed subset $\partial f(x)$ of $\mathbb{X}^* = \mathbb{X}$. To that set $\partial f(x)$ is associated an *abstract directional derivative* $f'(x; \cdot)$ presumed to satisfy

$$f'(x; v) \leq \sup \langle \partial f(x), v \rangle \text{ for all vectors } v,$$

with the convention $\sup \emptyset = +\infty$. For any function $\varphi : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$ its *outer Lebesgue integral* is defined by

$$\widehat{\int} \varphi := \inf \left\{ \int \hat{\varphi} : \varphi \leq \hat{\varphi} \text{ a.e. and } \hat{\varphi} \text{ Lebesgue integrable} \right\}.$$

Suppose that along any absolutely continuous trajectory $x(\cdot)$ that stays outside S on an interval $[\tau, t]$, the function f satisfies

$$f(x(t)) - f(x(\tau)) \leq \widehat{\int}_{[\tau,t]} f'(x; \dot{x}). \tag{15}$$

Let $f^+ := \max\{f, 0\}$ and posit $P(t, x) := \mu(t, x)\partial f^+(x)$, with $\mu(t, x) \geq 0$, to have (1) assume the form

$$\dot{x}(t) \in M(t, x(t)) - \mu(t, x(t))\partial f^+(x(t)). \tag{16}$$

We must however, be more specific as to which subgradients $g \in \partial f^+(x)$ will apply. Again motivated by steepest descent methods we insist that *while x stays infeasible any selection $g \in \partial f(x)$ must maximize the function* $g \mapsto \sup \langle \partial f(x), g \rangle$ whenever the maximum is attained. Further, for the efficiency of (16) we need to underestimate $\partial f(x)$ while x remains infeasible. Finally, for simpler statement, let $\langle A, B \rangle := \sup\{\langle a, b \rangle : a \in A, b \in B\}$.

Theorem 5. (Absorption in finite time and viability of sublevel sets) *With $\mathbb{X}$ Euclidean, suppose $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is lsc and $S := \{f \leq 0\}$ nonempty. Also suppose $f(x(0)) < \int_0^{T^-} \delta$ for some $T^- \in]0, T[$. Any solution $x(\cdot)$ to* (16) *on $[0, T[$, such that $f(x(0)) > 0$, and*

$$\langle \partial f(x), M(t, x) - \mu(t, x)\partial f(x) \rangle \leq -\delta \tag{17}$$

almost whenever $x = x(t) \notin S$, becomes permanently feasible no later than time

$$\hat{t} := \inf \left\{ t > 0 : f(x(0)) \leq \int_0^t \delta \right\}.$$

Proof. $x(\cdot)$ stays infeasible during for some time interval $[0, t]$. While $\tau \in [0, t]$, using simplified notations $x = x(\tau)$, $\mu = \mu(\tau, x)$, $\dot{x} = \dot{x}(\tau) = m - \mu g$ with $m \in M(\tau, x)$ and $g \in \partial f(x)$, we get

$$\begin{aligned} f'(x; \dot{x}) &\leq \sup \langle \partial f(x), \dot{x} \rangle \\ &\leq \sup \langle \partial f(x), M(x) - \mu \partial f(x) \rangle \leq -\delta. \end{aligned}$$

Now invoke (15) to get $f(x(0)) > \int_0^t \delta$, and the desired estimate obtains. If the sejour in S were transient, there would exist times $\tau > t > 0$ for which $0 < f(x(t)) < f(x(\tau))$ (since $x(\cdot)$ solves (16), $f(x(\tau)) < +\infty$). But along the arc $x(\cdot)$ that stretches from $x(t)$ forwards to $x(\tau)$, the inequality above implies the contradiction

$$0 < f(x(\tau)) - f(x(t)) \leq -\int_t^\tau \delta \leq 0. \quad \square$$

It suffices for (17) to have

$$\mu(t, x) d^2_{\partial f(x)}(0) \geq \delta(t) + \langle \partial f(x), M(t, x) \rangle \text{ when } x \notin S.$$

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